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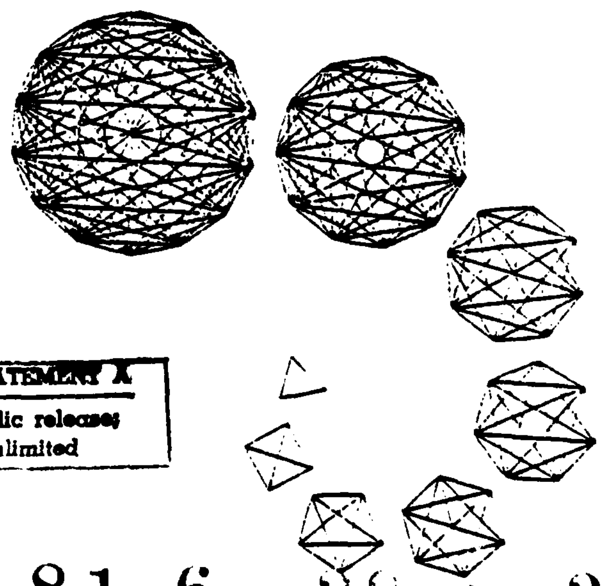

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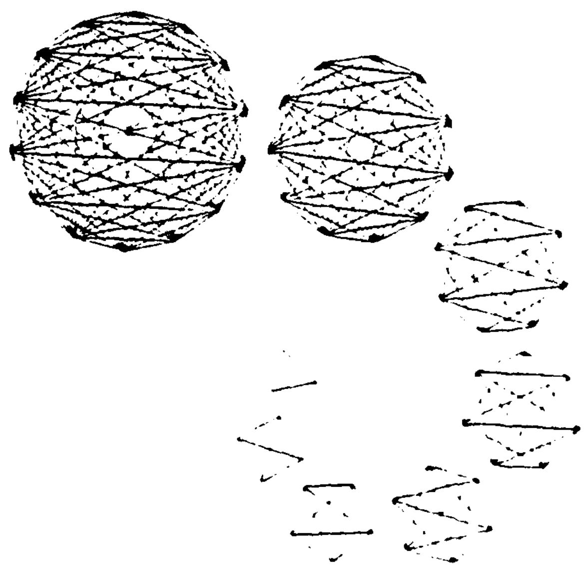

Technical Report #3

Symmetry and Bifurcation in Three Dimensional Elasticity, Part I

D.R.J. Chillingworth, J.E. Marsden and Y.H. Wan

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Solutions of their traction problem near natural state are completely classified and their stability determined.

SYMMETRY AND BIFURCATION IN THREE DIMENSIONAL ELASTICITY

PART I

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CONTENTS

GLOSSARY OF NOTATION

$\mathcal{B} \subset \mathbb{R}^3$	reference configuration
$T_X\mathcal{B}$	vectors in $\mathbb{R}^3$ based at $X \in \mathcal{B}$
$\phi: \mathcal{B} \to \mathbb{R}^3$, $x = \phi(X)$	deformation
$u: \mathcal{B} \to \mathbb{R}^3$	infinitesimal deformation
$e = [\nabla u + (\nabla u)^T]/2$	strain
$\mathcal{C}$	all deformations ϕ
$F = D\phi$	deformation gradient
F^T	transpose of F
$C = F^T F$	Cauchy Green tensor
W	Stored energy function
$P = \frac{\partial W}{\partial F}$	first Piola-Kirchhoff stress
$S = 2\frac{\partial W}{\partial C}$	*second Piola-Kirchhoff stress*
$A = \frac{\partial P}{\partial F}$	elasticity tensor
$\mathcal{C} = \frac{\partial S}{\partial C}$	(second) elasticity tensor
$c = 2\mathcal{C}\|_{\phi = I_{\mathcal{B}}}$	classical elasticity tensor
I or $I_{\mathcal{B}}$	identity map on $\mathbb{R}^3$ or $\mathcal{B}$
$\ell = (B, \tau)$	a (dead) load
$\mathcal{L}$	all loads with total force zero
$L(T_X\mathcal{B}, \mathbb{R}^3)$	all linear maps of $T_X\mathcal{B}$ to $\mathbb{R}^3$
$\mathrm{sym}(T_X\mathcal{B}, T_X\mathcal{B})$	symmetric linear maps of $T_X\mathcal{B}$ to $T_X\mathcal{B}$.
$SO(3)$	$\{Q \in L(\mathbb{R}^3, \mathbb{R}^3) \mid Q^T Q = I, \det Q = 1\}$
M_3	$L(\mathbb{R}^3, \mathbb{R}^3)$
sym	symmetric elements of M_3
skew = so(3)	skew symmetric elements of M_3
$\hat{v}$	infinitesimal rotation about the axis v
$\mathcal{L}_e$	equilibrated loads

$k: \mathcal{L} \to M_3$	astatic load map
$A = k(\ell)$	astatic load for a load ℓ
$j = (k\|(\ker k)^{\perp})^{-1}$	*non singular part of* k
$Skew = j(skew)$	skew viewed in load space
$Sym = j(sym)$	sym viewed in load space
$\Phi: \mathcal{C} \to \mathcal{L}$	$\Phi(\phi) = (-DIV\ P,\ P\cdot N)$
$\mathcal{C}_{sym}$	$\{u: \mathcal{B} \to \mathbb{R}^3 \| u(0) = 0,\ \nabla u(0) \in sym\}$
$\mathcal{N}$	image of $\mathcal{C}_{sym}$ near $I_{\mathcal{B}}$ under Φ
$F: \mathcal{L}_e \to Skew$	$\mathcal{N}$ is the graph of F
$\bar{F}: \mathbb{R} \times \mathcal{L}_e \to Skew$	$\bar{F}(\lambda,\ell) = F(\lambda\ell)/\lambda^2$
S_A	Q's in $SO(3)$ that equilibrate A.

<u>Editorial Note</u>: To avoid confusion of notation, vectors and tensors are boldface; vectors and 2-tensors are boldface italic and 4-tensors are in boldface block letters. All other mathematical symbols will be light-face italic.

§1. Introduction

The purpose of this paper is to study the traction problem in three dimensional nonlinear elasticity using geometric techniques and singularity theory. The first two papers in the series deal with dead loads and configurations that are nearly stress free. As was shown by Signorini [1930] and Stoppelli [1958], this problem has nontrivial solutions. However, their analysis is incomplete for three reasons. First, their load was varied only by a scalar factor; in a <u>full</u> neighborhood in load space of a load which has an axis of equilibrium there are additional solutions missed by their analysis. Second, their analysis is only local in the rotation group, so additional nearly stress free solutions are missed by restricting to rotations near the identity. Third, some classes of loads with a degenerate axis of equilibrium were not considered. This paper completes their analysis by treating these questions as well as stability. The complexity of the answer is indicated by the fact that near certain types of loads, we find up to 40 distinct solutions that are nearly stress free. Our constitutive hypotheses on the stress tensor are 'generic'; for a degenerate stress tensor there can be even more solutions.

The literature on this problem is very extensive, going back to Signorini in the 1930's. Our primary sources have been Stoppelli [1958], Grioli [1962], Truesdell and Noll [1965], Van Buren [1968], Wang and Truesdell [1973], and Capriz and Podio Guidugli [1974]. However, none of the references beyond Stoppelli [1958] proves any of the theorems dealing with nontrivial cases; i.e. loads with axes of equilibrium. However, Grioli [1962] is a convenient reference for the statements.

The outline of this first part of the paper is as follows. Our notation for nonlinear elasticity and the problem near a natural state is formulated in Section 2. In Section 3 the basic properties of the astatic load are reviewed and developed. The problem is reformulated with special reference to the global aspects of the rotation group in Section 4 and introduces the bifurcation equation which plays a crucial role throughout the paper. Section 5 treats loads with no axis of equilibrium; there are three new features in this section. First the proof of Stoppelli's results is considerably simplified. Second, the results are global relative to the rotation group. Finally, the stability of the solutions is determined. *The number of solutions* will be classified by load types; this classification scheme is explained in Section 6. (Some work related to the "type classification" was given by Ogden [1977]). In Section 7 a second order bifurcation equation is shown to be a gradient. This consequence of Betti reciprocity is basic to our analysis. Section 8 gives a complete bifurcation analysis of type 1 loads (the case considered by Stoppelli), including a stability analysis. New local and global solutions are found. The final section makes explicit the comparison with Stoppelli's theorem.

The second paper of this series will analyze the remaining types 2, 3 and 4, using a reformulation of our gradient results, discuss linearization stability, parallel loads and will give additional connections with the literature.

Acknowledgments

The Signorini-Stoppelli problem was first introduced to us by Robin Knops. Since then a number of people have made useful comments, including Stuart Antman, John Ball, Roger Brockett, Martin Golubitsky, David Schaeffer, Morton Gurtin, and Clifford Truesdell.

§2. Statement of the Problem

Let $\mathcal{B} \subset \mathbb{R}^3$ be an open bounded set with smooth boundary* and assume for convenience that $0 \in \mathcal{B}$. Let $1 < p < \infty$, $s > (3/p) + 1$ and let $\mathcal{C}$ be the space of maps $\phi\colon \bar{\mathcal{B}} \to \mathbb{R}^3$ that are of class $W^{s,p}$ (so they are C^1) such that $\phi(0) = 0$, ϕ is a $W^{s,p}$-diffeomorphism onto its image, and $J(\phi) > 0$, where $J(\phi)$ is the Jacobian of ϕ.

For example, if $\psi\colon \bar{\mathcal{B}} \to \mathbb{R}^3$ is $W^{s,p}$-close to the identity and $\psi(0) = 0$, then $\psi \in \mathcal{C}$. If Q is a linear isomorphism of $\mathbb{R}^3$ to $\mathbb{R}^3$ with $\det Q > 0$, then $Q \circ \psi \in \mathcal{C}$ as well.

Let points in $\bar{\mathcal{B}}$ be denoted $X \in \bar{\mathcal{B}}$ and points in $\mathbb{R}^3$ be denoted x. Sometimes we write $x = \phi(X)$. Let $T_X\mathcal{B}$ be the tangent space to $\mathcal{B}$ at X, regarded as vectors in $\mathbb{R}^3$ based at X. We do not identify $T_X\mathcal{B}$ and $\mathbb{R}^3$ for conceptual clarity. For $\phi \in \mathcal{C}$, let $F(X) \in L(T_X\mathcal{B}, \mathbb{R}^3)$ be the derivative of ϕ at X; by standard abuse of notation we write $F(X) = D\phi(X)$ or $\nabla\phi(X)$ interchangeably and $L(T_X\mathcal{B}, \mathbb{R}^3)$ denotes the set of all linear maps of $T_X\mathcal{B}$ to $\mathbb{R}^3$. We let $F(X)^T \in L(\mathbb{R}^3, T_X\mathcal{B})$ denote the adjoint of $F(X)$ relative to the Euclidean inner product. Observe that $F(X) \in L_+(T_X\mathcal{B}, \mathbb{R}^3)$, the linear transformations with positive determinant, since $\det F(X) = J(\phi)(X) > 0$. We let $C = F^TF$ (that is, $C(X) = F(X)^TF(X) \in L(T_X\mathcal{B}, T_X\mathcal{B})$) denote the Cauchy-Green tensor.

---

*We believe that our results also hold when $\mathcal{B}$ has piecewise smooth boundary. This program depends on elliptic regularity for such regions. Except in special cases, this theory is non-existent and seems to depend on a modification of the usual Sobolev spaces near corners. However, for simple shapes like cubes, the necessary regularity can be checked by hand in situations where the linearized elastostatic equations can be solved explicitly.

Observe that $C(X) \in \mathrm{sym}_{\mathrm{pos}}(T_X\mathcal{B},\, T_X\mathcal{B})$, the positive definite symmetric linear transformations on $T_X\mathcal{B}$.

Assume we are given a <u>smooth stored energy function</u> W defined on pairs (X,C) where $C \in \mathrm{sym}_{\mathrm{pos}}(T_X\mathcal{B},\, T_X\mathcal{B})$. For $\phi \in \mathcal{C}$, the <u>stored energy of</u> ϕ is $\int_{\mathcal{B}} W(X,C(X))\,dV(X)$, where C is the Cauchy-Green tensor of ϕ and dV is the volume element in $\mathcal{B}$. The fact that W depends on ϕ only through the point values of C is usually called material frame indifference. (See Truesdell and Noll [1965] and Marsden and Hughes [1978] for discussions.) Since C is a function of F, we shall, by abuse of notation also write $W(X,F)$ for $W(X,F^TF)$.

The first <u>Piola-Kirchhoff stress tensor</u> $P(X,F)$ is defined by $P(X,F) = \frac{\partial}{\partial F} W(X,F)$, the partial derivative of W with respect to F. Thus, $P(X,F) \in L(T_X\mathcal{B}, \mathbb{R}^3)^*$. The <u>second Piola-Kirchhoff stress tensor</u> $S(X,C)$ is defined by $S(X,C) = 2\frac{\partial}{\partial C} W(X,C)$, so $S(X,C) \in \mathrm{sym}(T_X\mathcal{B},\, T_X\mathcal{B})^*$. From the chain rule, one has the relationship

$$P(X,F)\cdot G = \frac{1}{2} S(X,C)\cdot [F^TG + G^TF]$$

for all $G \in L(T_X\mathcal{B}, \mathbb{R}^3)$.[†]

For finite dimensional vector spaces V,W, the bilinear map $L(W,V) \times L(V,W) \to \mathbb{R};\ (A,B) \mapsto \frac{1}{2}\mathrm{trace}(A\,B)$ defines an isomorphism $L(V,W)^* \cong L(W,V)$. If $V = W$ is an inner product space, this isomorphism identifies $\mathrm{sym}(V,V)^*$ with $\mathrm{sym}(V,V)$. Using this identification, we get $P(X,F) \in L(\mathbb{R}^3,\, T_X\mathcal{B})$, $S(X,C) \in \mathrm{sym}(T_X\mathcal{B}, T_X\mathcal{B})$ and $P(X,F) = S(X,C)\circ F^T$, or $P = SF^T$ for short.

---

†The mass density does not appear in our formulas as we are building it into the definitions and work, for example, with body force per unit volume rather than per unit mass.

Let $A(X,F) = \frac{\partial P}{\partial F}(X,F)$ denote the elasticity tensor. Identifying $L(L(T_X\mathcal{B},\mathbb{R}^3),\ L(T_X\mathcal{B},\mathbb{R}^3)^*)$ in the usual way for second derivatives with $B(L(T_X\mathcal{B},\mathbb{R}^3))$, the space of bilinear maps of $L(T_X\mathcal{B},\mathbb{R}^3) \times L(T_X\mathcal{B},\mathbb{R}^3)$ to $\mathbb{R}$, $A(X,F)$ determines a symmetric bilinear map on $L(T_X\mathcal{B},\mathbb{R}^3)$.

The second elasticity tensor $\mathcal{C}(X,C)$ is similarly defined to be $\frac{\partial S}{\partial C} = 2\frac{\partial^2 W}{\partial C\partial C}$ evaluated at (X,C), so may be regarded as a symmetric bilinear map on $\mathrm{sym}(T_X\mathcal{B},T_X\mathcal{B})$. The chain rule gives

$$2A(X,F)\cdot(G,H) = \mathcal{C}(X,C)\cdot(F^TH + H^TF,\ F^TG + G^TF) + S(X,C)\cdot(H^TG + G^TH)$$

The following two assumptions will be made in the first two papers of this series.

(H1) The undeformed state is stress free; i.e. $P(X,I) = 0$, or equivalently, $S(X,I) = 0$, where I is the identity.

(H2) Strong ellipticity holds: there is an $\varepsilon > 0$ such that

$$A(X,I)\cdot(v\otimes\xi,\ v\otimes\xi) \geq \varepsilon\|\xi\|^2\|v\|^2$$

for all $\xi \in T_X\mathcal{B}^*$ and $v \in \mathbb{R}^3$, where $v\otimes\xi \in L(T_X\mathcal{B},\mathbb{R}^3)$ is defined by $(v\otimes\xi)(V) = v\cdot\xi(V)$.

The classical elasticity tensor is defined by $\mathbb{c}(X) = 2\mathcal{C}(X,I)$, so $\mathbb{c}(X)$ is a symmetric bilinear mapping on $\mathrm{sym}(T_X\mathcal{B},T_X\mathcal{B})$ to $\mathbb{R}$; at $\phi = I$, we identify $T_X\mathcal{B}$ and $\mathbb{R}^3$ since x and X coincide. By (H1),

$$A(X,I)\cdot(G,H) = \frac{1}{4}\mathbb{c}(X)\cdot(G + G^T,\ H + H^T)\ .$$

Regarding $A(X,I) \quad L(L(T_X\mathcal{B},T_X\mathcal{B}),\ L(T_X\mathcal{B},\ T_X\mathcal{B}))$ and $c(X) \in L(\mathrm{sym}(T_X\mathcal{B},\ T_X\mathcal{B}),\ \mathrm{sym}(T_X\mathcal{B},\ T_X\mathcal{B}))$; this reads

$$2A(X,I)\cdot G = c(X)\cdot(G + G^T) \quad ,$$

or, if G is symmetric,

$$A(X,I)\cdot G = c(X)(G) \quad .$$

By (H2), solvability of the linearized equations of elastostatics can be determined by the Fredholm alternative (see, e.g., Marsden and Hughes [1978]).

We shall let $B\colon\ \mathcal{B} \to \mathbb{R}^3$ denote a given body force (per unit volume) and $\tau\colon\ \partial\mathcal{B} \to \mathbb{R}^3$ a given surface traction (per unit area). These are dead loads; in other words, the equilibrium equations for ϕ that we are studying are:

$$\text{(E)} \qquad \begin{cases} \mathrm{DIV}\ P(X,\ F(X)) + B(X) = 0 & \text{for } X \in \mathcal{B} \\ P(X,F(X))\cdot N(X) = \tau(X) & \text{for } X \in \partial\mathcal{B} \end{cases}$$

where $N(X)$ is the outward unit normal to $\partial\mathcal{B}$ at $X \in \partial\mathcal{B}$ and $\mathrm{DIV}\ P$ is the divergence* of $P(X,F(X))$ with respect to X.

Let L denote the space of all pairs $(B,\tau) = \ell$ of loads (of class $W^{s-2,p}$ on $\mathcal{B}$ and $W^{s-1-1/p,p}$ on $\partial\mathcal{B}$) such that

$$\int_{\mathcal{B}} B(X)\,dV(X) + \int_{\partial\mathcal{B}} \tau(X)\,dA(X) = 0$$

---

*Recall from above that $P(X,F(X)) \in L(\mathbb{R}^3,\ T_X\mathcal{B})$. For any $v \in \mathbb{R}^3$, then $P(X,F(X))v$ defines a vector field on $\mathcal{B}$; its divergence defines the vector field $\mathrm{DIV}\ P$ by $(\mathrm{DIV}\ P)\cdot v = \mathrm{DIV}\ (Pv)$.

i.e. the total force on $\mathcal{B}$ vanishes, where dV and dA are the respective volume and area elements on $\mathcal{B}$ and $\partial\mathcal{B}$. Using the divergence theorem, observe that if the pair (B,τ) is such that (E) holds for some $\phi \in C$, then $(B,\tau) \in L$.

Throughout the paper, the group $SO(3) = \{Q \in L(\mathbb{R}^3,\mathbb{R}^3) \mid Q^TQ = I$ and $\det Q = +1\}$ of proper orthogonal transformations will play a key role.

By (H1), $\phi = I_{\mathcal{B}}$ (the identity map on $\mathcal{B}$) solves (E) with $B = \tau = 0$. By material frame indifference, $\phi = Q|\mathcal{B}$ is also a solution for any $Q \in SO(3)$. The map $Q \mapsto Q|\mathcal{B}$ embeds $SO(3)$ into C and we shall identify its image with $SO(3)$. Thus, the "trivial" solutions of (E) are elements of $SO(3)$.

Our basic problem is as follows:

(P1) <u>Describe the set of all solutions of (E) near the trivial solutions</u> $SO(3)$ <u>for various loads</u> $\ell \in L$ <u>near zero</u>.

Here "describe" includes the following objectives:

(a) counting the solutions

(b) determining the stability of the solutions

(c) showing that the results are insensitive to small perturbations of the stored energy function and the load.

§3. The Astatic Load and Axes of Equilibrium

This section is devoted to the geometry of the load space L. Many of the results of this section are available in the literature, but we gather them here for convenience.

Before beginning, we shall recall a few notations and facts about the rotation group $SO(3)$. Let

$$
\begin{aligned}
M_3 = L(\mathbb{R}^3, \mathbb{R}^3) &= \text{linear transformations of } \mathbb{R}^3 \text{ to } \mathbb{R}^3 \\
\text{sym} &= \{A \in M_3 \mid A^T = A\} \\
\text{skew} &= \{A \in M_3 \mid A^T = -A\}
\end{aligned}
$$

We identify skew with $so(3)$, the Lie algebra of $SO(3)$; skew and $\mathbb{R}^3$ are isomorphic by the mapping $v \in \mathbb{R}^3 \mapsto \hat{v} \in \text{skew}$, where $\hat{v}(w) = w \times v$. If $v = (p,q,r)$ relative to the standard basis, then

$$
\hat{v} = \begin{bmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{bmatrix}
$$

The Lie bracket is $[\hat{v},\hat{w}] = v \otimes w - w \otimes v = -(v \times w)^{\wedge}$ where $v \otimes w \in M_3$ is given by $(v \otimes w)(u) = v\langle w,u\rangle$. The inner product is $\langle v,w\rangle = \frac{1}{2}\text{trace}\,(\hat{v}^T \hat{w})$, the Killing form on $so(3)$. Finally, $\exp(\hat{v})$ is the rotation about the vector v in the positive sense, through the angle $\|v\|$.

Now we turn to a study of L. For $\phi \in C$ and $\ell \in L$, we say that ℓ is equilibrated relative to ϕ if the total torque in the configuration ϕ vanishes:

$$\int_{\mathcal{B}} \phi(X) \times B(X)dV(X) + \int_{\partial\mathcal{B}} \phi(X) \times \tau(X)dA(X) = 0$$

where $\ell = (B,\tau)$. From symmetry of the stress one sees that if $\ell = (B,\tau) \in \mathcal{L}$ satisfies (E) for some $\phi \in \mathcal{C}$, then ℓ is equilibrated relative to ϕ. (An easy proof uses the Piola transform; cf. Marsden and Hughes [1978]).

Let $\mathcal{L}_e$ denote the loads that are equilibrated relative to the identity configuration $I_{\mathcal{B}}$.

Define the <u>astatic load map</u> $k: \mathcal{L} \times \mathcal{C} \to M_3$ by

$$k(\ell,\phi) = \int_{\mathcal{B}} B(X) \otimes \phi(X)dV(X) + \int_{\partial\mathcal{B}} \tau(X) \otimes \phi(X)dA(X)$$

and write $k(\ell) = k(\ell, I_{\mathcal{B}})$.

We have actions of $SO(3)$ on $\mathcal{L}$ and $\mathcal{C}$ given by:

<u>Action of</u> $SO(3)$ <u>on</u> $\mathcal{L}$: $Q\ell(X) = (QB(X), Q\tau(X))$

<u>Action of</u> $SO(3)$ <u>on</u> $\mathcal{C}$: $Q\phi = Q\circ\phi$

Note that $Q\ell$ means "the load arrows are rotated, keeping the body fixed." We shall write $\mathcal{O}_\ell$ and $\mathcal{C}_\phi$ for the $SO(3)$ orbits of ℓ and ϕ. Thus, $\mathcal{O}_{I_{\mathcal{B}}}$ denotes the trivial solutions corresponding to $\ell = 0$.

The following is a list of basic observations about the astatic load, each of which is readily verified:

(A1) ℓ <u>is equilibrated relative to</u> ϕ <u>if and only if</u> $k(\ell,\phi) \in$ sym. <u>In particular,</u> $\ell \in L_e$ <u>if and only if</u> $k(\ell) \in$ sym.

(A2) <u>(equivariance)</u>. <u>For</u> $\ell \in L$, $\phi \in C$, <u>and</u> $Q_1, Q_2 \in SO(3)$

$$k(Q_1\ell, Q_2\phi) = Q_1 k(\ell,\phi) Q_2^{-1}$$

<u>In particular,</u> $k(Q\ell) = Qk(\ell)$

(A3) <u>(infinitesimal equivariance)</u>. <u>For</u> $\ell \in L$, $\phi \in C$, <u>and</u> W_1, $W_2 \in$ skew,

$$k(W_1\ell,\phi) = W_1 k(\ell,\phi),\ k(\ell, W_2\phi) = -k(\ell,\phi)\ W_2$$

<u>In particular</u>, $k(W\ell) = Wk(\ell)$.

Later on, we shall be concerned with how the orbit of $\ell \in L$ meets L_e. The most basic result in this direction is the following.

3.1. <u>DaSilva's Theorem</u>. <u>Let</u> $\ell \in L$. <u>Then</u> $O_\ell \cap L_e \neq \emptyset$.

<u>Proof</u>. By the polar decomposition, we can write $k(\ell) = Q^T A$ for some $Q \in SO(3)$ and $A \in$ sym. By (A2), $k(Q\ell) = Qk(\ell) = A \in$ sym, so by (A1), $Q\ell \in L_e$. ■

Similarly, any load can be equilibrated relative to any chosen configuration by a suitable rotation.

The concept of an axis of equilibrium deals with the case in which O_ℓ meets L_e in a degenerate way.

3.2. <u>Definition</u>. Let $\ell \in L_e$ and $v \in \mathbb{R}^3$, $\|v\| = 1$. We say that v is an <u>axis of equilibrium for</u> ℓ when $\exp(\theta\hat{v})\ell \in L_e$ for all real θ, i.e. when rotations of ℓ about the axis v do not destroy equilibration relative to the identity.

A number of useful ways of reformulating the condition that v be an axis of equilibrium are as follows.

3.3. <u>Proposition</u>. <u>Let</u> $\ell \in L_e$ <u>and</u> $A = k(\ell) \in \text{sym}$. <u>The following conditions are equivalent</u>:

1. ℓ <u>has an axis of equilibrium</u> v
2. <u>there is a</u> $v \in \mathbb{R}^3$, $\|v\| = 1$ <u>such that</u> $\hat{v}\ell \in L_e$
3. $W \mapsto AW + WA$ <u>fails to be an isomorphism of skew to itself</u>
4. <u>trace</u> A <u>is an eigenvalue of</u> A.

<u>Proof</u>. $1 \Rightarrow 2$. This follows by differentiating $\exp(\theta\hat{v})\ell$ in θ at $\theta = 0$. $2 \Rightarrow 1$. By (A2),

$$k(\exp(\theta\hat{v})\ell) = [I + (\theta\hat{v}) + \frac{1}{2!}(\theta\hat{v})^2 + \ldots]k(\ell) .$$

Since $k(\hat{v}\ell) = \hat{v}k(\ell)$ is symmetric, this is symmetric, term by term.

$2 \Rightarrow 3$. Since $k(\hat{v}\ell) = \hat{v}A$ is symmetric, $\hat{v}A + A\hat{v} = 0$, so $W \mapsto AW + WA$ is not an isomorphism.

$3 \Rightarrow 2$. There exists a $v \in \mathbb{R}^3$, $\|v\| = 1$, such that $\hat{v}A + A\hat{v} = 0$, so $k(\hat{v}\ell) = \hat{v}A$ is symmetric.

$3 \Leftrightarrow 4$. Define $L \in M_3$ by $L = (\text{trace } A)I - A$. Then one has the relationship

$$(Lv)^\wedge = A\hat{v} + \hat{v}A$$

(In fact, if $[u,v,w]$ denotes the triple product, the relationship $[Bu,Bv,Bw] = \det B[u,v,w]$ gives $[Au,v,w] + [u,Av,w] + [u,v,Aw] = (\text{trace } A)\,[u,v,w]$. This yields $(Lv)^\wedge = \hat{v}A + A^T\hat{v}$, which gives the claimed results for symmetric A). Therefore, $A\hat{v} + \hat{v}A = 0$ if and only if $Lv = 0$, i.e., v is an eigenvector of A with eigenvalue trace A . ■

3.4. Corollary. Let $\ell \in L_e$ and $A = k(\ell) \in \text{sym}$. Let the eigenvalues of A be denoted a, b, c. Then ℓ has no axis of equilibrium if and only if $(a + b)(a + c)(b + c) \neq 0$.

Proof. This condition is equivalent to saying that trace A is not an eigenvalue of A. ■

3.5. Definition. We shall say that $\ell \in L_e$ is a type 0 load if ℓ has no axis of equilibrium and if the eigenvalues of $A = k(\ell)$ are distinct.

The following shows how the orbits of type 0 loads meet L_e.

3.6. Proposition. Let $\ell \in L_e$ be a type 0 load. Then $\mathcal{O}_\ell \cap L_e$ consists of four type 0 loads.

Proof. We first prove that the $SO(3)$-orbit of A in M_3 under the action $Q \mapsto QA$ meets sym in four points. Relative to its basis of eigenvectors, we can write $A = \text{diag}(a,b,c)$. Then $\mathcal{O}_A \cap \text{sym}$ contains the four points

$$\begin{array}{ll} \operatorname{diag}(a,b,c) & (Q = I) \\ \operatorname{diag}(-a,-b,c) & (Q = \operatorname{diag}(-1,-1,1)) \\ \operatorname{diag}(-a,b,-c) & (Q = \operatorname{diag}(-1,1-1)) \\ \operatorname{diag}(a,-b,-c) & (Q = \operatorname{diag}(1,-1,-1)) \end{array}$$

These are distinct points since $(a + b)(a + c)(b + c) \neq 0$. Now suppose a, b and c are distinct. Suppose $QA = S \in \mathrm{sym}$. Then $S^2 = A^2$. Let μ_i be an eigenvalue of S with eigenvector u_i. Then $S^2 u_i = \mu_i^2 u_i = A^2 u_i$, so μ_i^2 is an eigenvalue of A^2. Thus, as the eigenvectors of A^2 with a given eigenvalue are unique, u_i is an eigenvector of A and $\pm\mu_i$ is the corresponding eigenvalue. Since $\det Q = +1$, $\det S = \det A$, so we must have one of the four cases above.

By equivariance, $k(\mathcal{O}_\ell) \cap \mathrm{sym} = \mathcal{O}_{k(\ell)} \cap \mathrm{sym}$ is a set consisting of four points. Now $\mathcal{O}_\ell \cap L_e = k^{-1}(\mathcal{O}_{k(\ell)} \cap \mathrm{sym})$, so it suffices to show that k is one-to-one on $\mathcal{O}_\ell$. This is a consequence of the following and (A2).

3.7. Lemma. *Suppose* $A \in \mathrm{sym}$ *and* $\dim \ker A \leqslant 1$. *Then* A *has no isotropy; i.e.* $QA = A$ *implies* $Q = I$.

Proof. Every $Q \neq I$ acts on $\mathbb{R}^3$ by rotation through an angle θ about a unique axis, that is, about a line through the origin in $\mathbb{R}^3$. Now $QA = A$ means that Q is the identity on the range of A. Therefore if $Q \neq I$ and $QA = A$, the range of A must be 0 or 1 dimensional, i.e. $\dim \ker A \geqslant 2$. ∎

Finally in this section, we study the range and kernel of $k: L \to M_3$

3.8. Proposition. 1. ker k consists of those loads in L_e for which every axis is an axis of equilibrium.

2. $k: L \to M_3$ is surjective.

Proof. 1. Let $\ell \in \ker k$. For $W \in$ skew, $k(W\ell) = Wk(\ell) = 0$ so $W\ell \in L_e$; by 3.3 every axis is an axis of equilibrium. Conversely, if $W\ell \in L_e$ for all $W \in$ skew, then $k(W\ell) = Wk(\ell)$ is symmetric for all W; i.e. $k(\ell)W + Wk(\ell) = 0$ for all W. From $(Lv)^\wedge = A\hat{v} + \hat{v}A$, where $A = k(\ell)$, and $L = (\text{trace } A)I - A$, we see that $L = 0$. This implies trace $A = 0$ and hence $A = 0$.

To prove 2 introduce the SO(3) invariant inner product on L:

$$\langle \ell, \bar{\ell} \rangle = \int_B \langle B(X), \bar{B}(X) \rangle dV(X) + \int_{\partial B} \langle \tau(X), \bar{\tau}(X) \rangle dA(X).$$

Relative to this and the inner product $\text{trace}(A^T B)/2$ on M_3, the adjoint $k^T: M_3 \to L$ of k is given by

$$k^T(A) = (B, \tau) \quad \text{where} \quad B(X) = AX - G, \ \tau(X) = AX - G,$$

and

$$G = \left[\int_B AX \, dV(X) + \int_{\partial B} AX \, dA(X)\right] \Big/ \left[\int_B dV + \int_{\partial B} dA\right].$$

If $k^T(A) = (0,0)$ then it is clear that $A = 0$. It follows that k is surjective. ■

3.9. Corollary. 1. $\ker k$ is the largest subspace of $\mathcal{L}_e$ that is $SO(3)$ invariant.

2. $k|(\ker k)^{\perp} : (\ker k)^{\perp} \to M_3$ is an isomorphism.

Let $j = (k|(\ker k)^{\perp})^{-1}$ and write

$$\text{Skew} = j(\text{skew}) \quad \text{Sym} = j(\text{sym})$$

These are linear subspaces of $\mathcal{L}$ of dimension 3 and 6 respectively. Thus we have the decomposition:

$$\mathcal{L} = \overbrace{\text{Skew} \oplus \text{Sym}}^{SO(3)\text{ invariant pieces}} \oplus \underbrace{\overbrace{\ker k}}_{\mathcal{L}_e}$$

corresponding to the decomposition $M_3 = \text{skew} \oplus \text{sym}$;

$U = \frac{1}{2}(U - U^T) + \frac{1}{2}(U + U^T)$.

Note: Skew and $\mathcal{L}_e$ need not be orthogonal.

§4. Equivalent Reformulations of the Problem

Define $\Phi: \mathcal{C} \to \mathcal{L}$ by $\Phi(\phi) = (-\mathrm{DIV}\, P,\ P\cdot N)$ i.e.

$$\Phi(\phi)(X) = (-\mathrm{DIV}\, P(X,F(X)),\ P(X,F(X))\cdot N(X))$$

so the equilibrium equations (E) become $\Phi(\phi) = \ell$. From material frame indifference we have equivariance of Φ; i.e. $\Phi(Q\phi) = Q\Phi(\phi)$. Standard Sobolev estimates show that Φ is a smooth mapping (see, for example, Palais [1968]). The derivative of Φ is given by

$$D\Phi(\phi)\cdot u = (-\mathrm{DIV}(A\cdot\nabla u),(A\cdot\nabla u)\cdot N)$$

and at $\phi = I_{\mathcal{B}}$ this becomes

$$D\Phi(I_{\mathcal{B}})\cdot u = (-\mathrm{DIV}(\mathcal{C}\cdot e),(\mathcal{C}\cdot e)\cdot N)$$

where $e = \frac{1}{2}[\nabla u + (\nabla u)^T]$.

If $D\Phi(I_{\mathcal{B}}): T_{I_{\mathcal{B}}}\mathcal{C} \to \mathcal{L}$ were an isomorphism we could solve $\Phi(\phi) = \ell$ uniquely for ϕ near $I_{\mathcal{B}}$ and ℓ small. The essence of our problem is that $D\Phi(I_{\mathcal{B}})$ is <u>not</u> an isomorphism; since $\Phi(SO(3))=0$, kernel $D\Phi(I_{\mathcal{B}})$ contains skew.

Define $\mathcal{C}_{sym} = \{u \in T_{I_{\mathcal{B}}}\mathcal{C} \mid u(0) = 0$ and $\nabla u(0) \in sym\}$. From (H2) and linear elasticity, we have:

4.1. <u>Lemma</u>. $D\Phi(I_{\mathcal{B}})|\mathcal{C}_{sym}: \mathcal{C}_{sym} \to \mathcal{L}_e$ <u>is an isomorphism.</u>

The connection between the astatic load map $k: \mathcal{L} \to M_3$ and Φ is seen from the following computation of $k \circ \Phi$.

4.2. Lemma. Let $\phi \in \mathcal{C}$ and let P be the first Piola-Kirchhoff stress tensor of ϕ. Then

$$k(\Phi(\phi)) = \int_{\mathcal{B}} p\,dV$$

This follows by an application of Gauss' theorem to

$$k(\Phi(\phi)) = \int_{\mathcal{B}} (-\mathrm{DIV}\ P) \otimes X\ dV(X) + \int_{\partial\mathcal{B}} (P\ N) \otimes X\ dA(X).$$

This should be compared with the astatic load relative to the configuration ϕ rather than $I_{\mathcal{B}}$; one gets

$$k(\Phi(\phi), \phi) = \int_{\mathcal{B}} S\ dV$$

which is symmetric, while $k(\Phi(\phi)) = k(\Phi(\phi), I_{\mathcal{B}})$ need not be.

To study solutions of $\Phi(\phi) = \ell$ for ϕ near the trivial solutions and ℓ near a given load ℓ_0, it suffices to take $\ell_0 \in \mathcal{L}_e$. This follows from Da Silva's theorem and equivariance of Φ.

Let $\mathcal{C}_{sym}$ be regarded as an affine subspace of $\mathcal{C}$ centered at $I_{\mathcal{B}}$. Let $\tilde{\Phi}$ be the restriction of Φ to $\mathcal{C}_{sym}$. From the implicit function theorem we get:

4.3. Lemma. There is a ball centered at I_B in C_{sym} whose image N under $\tilde{\Phi}$ is a smooth submanifold of L tangent to L_e at 0 (see Figure 1). The manifold N is the graph of a unique smooth mapping

$$F: L_e \to \text{Skew}$$

such that $F(0) = 0$ and $DF(0) = 0$.

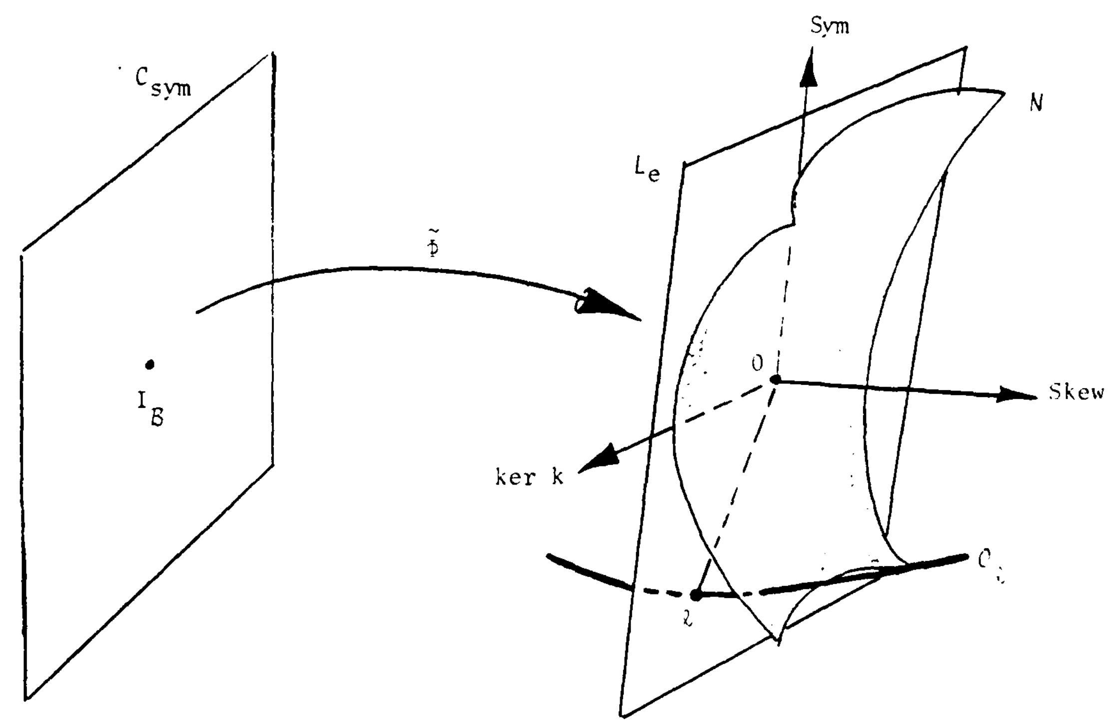

Figure 1

Later we shall show how to compute $D^2F(0)$ in terms of $D\Phi(I_B)^{-1}$ and c.

Now we are ready to reformulate problem (P1).

(P2) For a given $\ell_0 \in L_e$ near zero, study how O_ℓ meets the graph of F for various ℓ near ℓ_0.

Problems (P1) and (P2) are related as follows. Let ϕ solve (E) with $\ell \in L$ and Q be such that $\bar{\phi} = Q\phi \in C_{sym}$. Then $\Phi(\bar{\phi}) = Q\ell$, so the orbit of ℓ meets the graph of F at $\Phi(\bar{\phi})$. Conversely, if the orbit of ℓ meets N at $\Phi(\bar{\phi}) = Q\ell$, then $\phi = Q^{-1}\bar{\phi}$ solves (E). We claim that near the trivial solutions, the numbers of solutions to each problem also correspond. This follows from the next lemma.

4.4. Lemma. There is a neighborhood U of I_B in C_{sym} such that $\phi \in U$ and $Q\phi \in U$ implies $Q = I_B$.

Proof. Note that C_{sym} is transverse to O_{I_B} at I_B and I_B has trivial isotropy. Thus, as $SO(3)$ is compact, O_{I_B} is closed, so there is a neighborhood of U_0 of I_B in C_{sym} such that $Q|B \in U_0$ implies $Q = I$. The same thing is true of orbits passing through a small neighborhood of I_B by openness of transversality and compactness of $SO(3)$. ■

If O_ℓ meets N in k points $Q_i\ell = \Phi(\bar{\phi}_i)$, $i = 1,\ldots,k$ then $\bar{\phi}_i$ are distinct as Φ is 1-1 on a neighborhood of I_B in C_{sym}. If this neighborhood is also contained in U of 4.4, then the points

$Q_i^{-1}\bar{\phi}_i = \phi_i$ are also distinct by 4.4. Hence the problems (P1) and (P2) are equivalent.

In connection with the action $(Q,A) \mapsto QA$ of $SO(3)$ on M_3 we shall require some more notation. Let

$$\mathrm{Skew}(A) = \frac{1}{2}(A - A^T) \in \mathrm{skew} \tag{3.2a}$$

and

$$\mathrm{Sym}(A) = \frac{1}{2}(A + A^T) \in \mathrm{sym} \tag{3.2b}$$

be the skew symmetric and symmetric parts of A, respectively.

We shall, by abuse of notation, suppress j and identify *Sym* with sym and Skew with skew. Thus we will write a load $\ell \in L$ as $\ell = (A,n)$ where $A = k(\ell) \in M_3$ and $n \in \ker k$; hence $\ell \in L_e$ precisely when $A \in \mathrm{sym}$. The action of $SO(3)$ on L is given by

$$Q\ell = (QA,Qn).$$

Using this notation we can reformulate problem (P2) as follows:

(P3) <u>For a given</u> $\ell_0 = (A_0,n_0) \in L_e$ <u>near zero</u>, <u>and</u> $\ell = (A,n)$ <u>near</u> ℓ_0, <u>find</u> $Q \in SO(3)$ <u>such that</u>

$$\mathrm{Skew}(QA \quad F(\mathrm{Sym}(QA),Qn) = 0.$$

Define the rescaled map $\bar{F}: \mathbb{R} \times L_e \to \mathrm{Skew}$ by

$$\bar{F}(\lambda,\ell) = \frac{1}{\lambda^2}F(\lambda\ell).$$

Since $F(0) = 0$ and $DF(0) = 0$, $\bar{F}$ is smooth. Moreover, if $F(\ell) = \frac{1}{2}G(\ell) + \frac{1}{6}C(\ell) + \ldots$ is the Taylor expansion of F about zero, then $\bar{F}(\lambda,\ell) = \frac{1}{2}G(\ell) + \frac{\lambda}{6}C(\ell) + \ldots$.

In problem (E) let us measure the size of ℓ by the parameter λ. Thus, replace $\Phi(\phi) = \ell$ for ℓ near zero by $\Phi(\phi) = \lambda\ell$ for λ near zero. This scaling enables us to conveniently distinguish the size of ℓ from its 'orientation'. In the literature ℓ has always been fixed and λ taken small. Here we allow ℓ to vary as well. Thus we arrive at the final formulation of the problem.

(P4) <u>For a given</u> $\ell_0 = (A_0,n_0) \in L_e$, <u>for</u> ℓ <u>near</u> ℓ_0 <u>and</u> λ <u>small</u>, <u>find</u> $Q \in SO(3)$ <u>such that</u>

$$\mathrm{Skew}(QA) - \lambda\bar{F}(\lambda, \mathrm{Sym}(QA), Qn) = 0.$$

The left hand side of this equation will be denoted $H(\lambda,A,n;Q)$ or $H(\lambda,Q)$ if A, n are fixed.

§5. Type 0: No axis of equilibrium

We shall begin the analysis by giving an (almost trivial) proof of one of the basic theorems of Stoppelli [1958]*

5.1. Theorem. Suppose $\ell \in L_e$ has no axis of equilibrium. Then for λ sufficiently small, there is a unique $\tilde{\phi} \in C_{sym}$ and a unique Q in a neighborhood of the identity in $SO(3)$ such that $\phi = Q^{-1}\tilde{\phi}$ solves the traction problem

$$\Phi(\phi) = \lambda\ell .$$

Proof. Define $H: \mathbb{R} \times SO(3) \rightarrow \text{Skew}$ by

$$H(\lambda,Q) = \text{Skew}(QA) - \lambda\bar{F}(\lambda,\text{Sym}(QA),Qn))$$

where $\ell = (A,n) \in L_e = \text{Sym} \oplus \ker k$ is fixed. Note that $D_2H(0,I)\cdot W = \text{Skew}(WA) = \frac{1}{2}(WA + AW)$. By Proposition 3.3, this is an isomorphism. Hence, by the implicit function theorem, $H(\lambda,Q) = 0$ can be uniquely solved for Q near $I \in SO(3)$ as a function of λ near $0 \in \mathbb{R}$. ■

The geometric reason "why" this proof works and the clue to treating other cases is the following.

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*The only other complete proof in English we know of is given in Van Buren [1968], although sketches are available in Grioli [1963], Truesdell and Noll [1965] and Wang and Truesdell [1973]. Our proof is rather different; the use of the map $\bar{F}$ avoids a series of complicated estimates used by Stoppelli and Van Buren.

5.2 Lemma. A load $\ell \in L_e$ has no axis of equilibrium precisely when $L = L_e \oplus T_\ell \mathcal{O}_\ell$. In particular, if ℓ has no axis of equilibrium, then $\mathcal{O}_\ell$ intersects L_e transversely at ℓ.

Proof. The tangent space to $\mathcal{O}_\ell$ at $\ell \in L_e$ is $T_\ell \mathcal{O}_\ell = \{W\ell \mid W \in \text{skew}\}$, and the projection of this into the complement Skew to L_e is $W\ell \mapsto \frac{1}{2}(WA + AW)$ where $A = k(\ell)$. The result then follows from part 3 of 3.3. ■

We have shown that there is only one solution to $\Phi(\varphi) = \lambda\ell$ near the identity if λ is small and ℓ has no axis of equilibrium. How many solutions are there near the trivial solutions SO(3)? As we shall see, this problem has a non-trivial answer which depends on the type of ℓ. We analyze the simplest case here. Recall from definition 3.5 that a load $\ell \in L_e$ is said to be of type 0 if ℓ has no axis of equilibrium and if $A = k(\ell)$ has distinct eigenvalues.

Loads with no axis of equilibrium occur amongst other types of loads classified in the next section, and Stoppelli's theorem 5.1 applies to them. However, the global structure of the solutions ("global" being relative to SO(3)) is different for the different types. For type 0 the situation is as follows.

5.3. Theorem. Let $\ell_0 \in L_e$ be of type 0. Then for λ sufficiently small, $\Phi(\varphi) = \lambda\ell_0$ has exactly four solutions $\varphi_1, \varphi_2, \varphi_3$ and φ_4 in a neighborhood of the trivial solutions $SO(3) \subset \mathcal{C}$ (see Figure 2).

<u>Proof</u>. By 3.6, $\mathcal{O}_{\lambda\ell_0}$ meets L_e in four points. By 5.1, in a neighborhood of 0 in L, $\mathcal{O}_{\lambda\ell_0}$ meets N in exactly four points, the images of $\bar{\Phi}_1$, $\bar{\Phi}_2$, $\bar{\Phi}_3$ and $\bar{\Phi}_4$, say. Thus problem (P2) has four solutions. By the equivalence of (P1) and (P2), so does (P1). ■

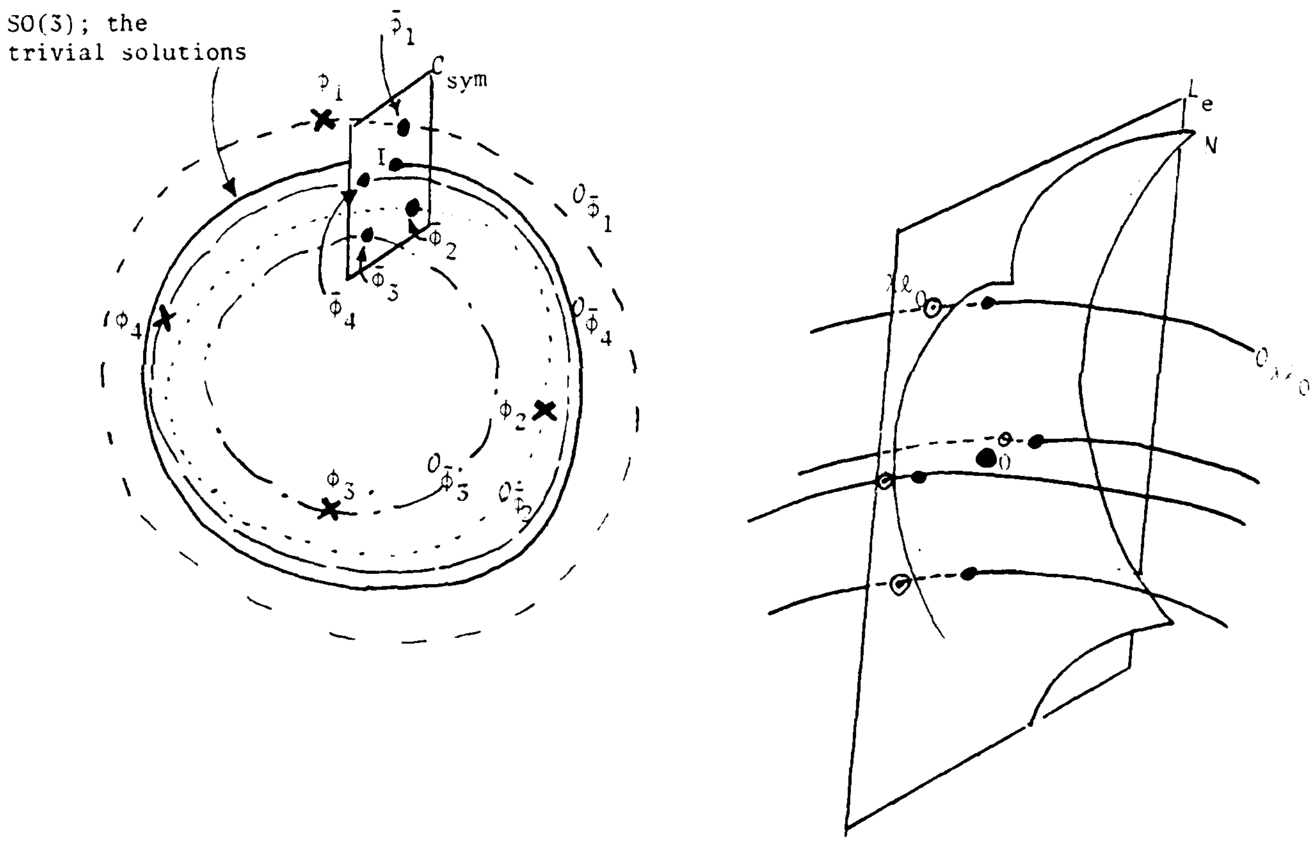

Figure 2

Let $A = k(\ell_0)$ and $S_A = \{Q \mid QA \in \text{sym}\}$. From the proof of 3.6 we see that S_A is a four element subgroup of $SO(3)$ isomorphic to $\mathbf{Z}_2 \oplus \mathbf{Z}_2$. By our earlier discussions, the elements ϕ_i are obtained from $\bar{\phi}_i$ by applying rotations close to elements of S_A. In particular, as $\lambda \to 0$, the solutions $\{\phi_i\}$ converge to the four element set S_A (regarded as a subset of C).

For ℓ sufficiently close to ℓ_0, the problem $\Phi(\phi) = \lambda\ell$ will also have four solutions. Indeed by the openness of transversality, $\mathcal{O}_{\lambda\ell}$ will also meet N in four points. In other words, the picture for type 0 in Figure 2 is stable under small perturbations of ℓ_0.

Next we study the stability of the four solutions found in Theorem 5.3. This will be done under the hypothesis that the classical elasticity tensor in stable. We introduce the following condition:

(H3) <u>Assume there is an</u> $\eta > 0$ <u>such that for all</u> $e \in \text{Sym}(T_X\mathcal{B}, T_X\mathcal{B})$,

$$\varepsilon(e) = \frac{1}{2}c(X)(e,e) \geqslant \eta\|e\|^2, \quad \|\cdot\| = \text{pointwise norm}$$

($\varepsilon(e)$ is the stored energy function for linearized elasticity).

Because of difficulties with potential wells and dynamical stability in elasticity (see Knops and Wilkes [1972] and Ball, Knops and Marsden [1978]) we shall adopt the following "energy criterion" definition of stability.

5.4. <u>Definition</u>. A solution ϕ of $\Phi(\phi) = \ell$ will be called <u>stable</u> if ϕ is a local minimum in C of the potential function

$$V_\ell(\phi) = \int_{\mathcal{B}} W(\phi)\,dV - \langle \ell, \phi \rangle$$

where $\langle \ell, \phi \rangle = \int_{\mathcal{B}} B(X)\cdot\phi(X)\,dV(X) + \int_{\partial\mathcal{B}} \tau(X)\cdot\phi(X)\,dA(X) = \text{trace}\; k(\ell,\phi)$.

If ϕ is not stable, its <u>index</u> is the dimension of the largest subspace of vectors u tangent to $\mathcal{C}$ at ϕ with the property that ϕ decreases along some curve tangent to u. (Thus, index 0 corresponds to stability.)

5.5. <u>Theorem</u>. <u>Assume (H1)-(H3) and let</u> ℓ_0 <u>be as in 5.3. For</u> λ <u>sufficiently small, amongst the four solutions</u> $\phi_1, \phi_2, \phi_3, \phi_4$ <u>given by 5.3, exactly one is stable; the others have indices</u> 1, 2, and 3. <u>Suppose</u> ϕ <u>is a solution approaching</u> $Q \in S_A$ <u>as</u> $\lambda \to 0$. <u>Then</u> ϕ <u>is stable if and only if</u> $QA - \text{trace}(QA)I \in \text{sym}$ <u>is positive-definite</u>. <u>In general, the index of</u> ϕ <u>is the number of negative eigenvalues</u> <u>of</u> $QA - \text{trace}\,(QA)I$.

<u>Proof</u>. Let $\phi_0 \in \mathcal{C}$ solve $\Phi(\phi) = \lambda\ell_0 = \ell$. Then ϕ_0 is a critical point of $V_{\lambda\ell_0}$. Consider the orbit $\mathcal{O}_{\phi_0} = \{Q\phi_0 \mid Q \in SO(3)\}$ of ϕ_0. Its tangent space decomposes $T_{\phi_0}\mathcal{C}$ as follows:

$$T_{\phi_0}\mathcal{C} = T_{\phi_0}\mathcal{O}_{\phi_0} \oplus (T_{\phi_0}\mathcal{O}_{\phi_0})^\perp$$

First consider V_ℓ restricted to $(T_{\phi_0}\mathcal{O}_{\phi_0})^\perp$. Its second derivative at ϕ_0 in the direction of $u \in (T_{\phi_0}\mathcal{O}_{\phi_0})^\perp$ is $\int_{\mathcal{B}} \frac{\partial^2 W}{\partial F\,\partial F}(\nabla\phi_0)\cdot(\nabla u, \nabla u)\,dV$. At

$\phi_0 = Q|\mathcal{B}$, this becomes $\int_{\mathcal{B}} c(X)\cdot(e(X),e(X))dV(X)$, where $e = \frac{1}{2}(\nabla u + (\nabla u)^T)$. This is larger than a positive constant times the L^2 norm of e, by (H3). However, since u is in $(T_\phi \mathcal{O}_\phi)^\perp$, $\|e\|^2_{L^2} \geqslant (\text{constant})\|u\|^2_{H^1}$ by Korn's inequality (see Fichera [1972]). By continuity, we have

$$D^2V_{\lambda\ell_0}(\phi_0)\cdot(u,u) \geqslant \delta\|u\|^2_{H^1}$$

if u is orthogonal to $\mathcal{O}_{\phi_0}$ at ϕ_0 and λ is small. This implies that ϕ_0 is a minimum for $V_{\lambda\ell_0}$ in directions transverse to $\mathcal{O}_{\phi_0}$. (Actually one can see that ϕ_0 is a local minimum in the topology of $\mathcal{C}$ on $(T_{\phi_0}\mathcal{O}_{\phi_0})^\perp$ by using Tromba's [1976] version of the Morse lemma.)

Next, consider $V_{\lambda\ell_0}$ restricted to $\mathcal{O}_{\phi_0}$. By material frame indifference, W is constant on $\mathcal{O}_{\phi_0}$ and so as ϕ_0 must be a critical point for $V_{\lambda\ell_0}$ restricted to $\mathcal{O}_{\phi_0}$, it is also a critical point for $\lambda\ell_0 = \ell$ restricted to $\mathcal{O}_{\phi_0}$ (where $\ell(\phi) = \langle \ell,\phi\rangle$). It suffices therefore to determine the index of $\ell|\mathcal{O}_{\phi_0}$ at ϕ_0. The result is now a consequence of continuity and the limiting case $\lambda \to 0$ given in the following lemma about type 0 loads.

5.6. Lemma. Let ℓ be type 0 and let $A = k(\ell)$. Then S_A, regarded as a subset of $\mathcal{C}$ equals the set of critical points of $\ell|\mathcal{O}_{I_{\mathcal{B}}}$. These 4 critical points are nondegenerate with indices 0, 1, 2 and 3; the index of Q is the number of negative eigenvalues of QA-trace$(QA)I$.

<u>Proof</u>. First note that $L_e = (T_{I_B} SO(3))^{\perp}$ since $D\Phi(I_B)$ has kernel $T_{I_B} SO(3)$ = skew, has range L_e and is self-adjoint. Thus $Q\ell \in L_e$ if and only if $\ell \perp T_{Q^T} SO(3)$. It follows that $Q\ell \in L_e$ if and only if Q^T is a critical point of $\ell | \mathcal{O}_{I_B}$. (Recall that elements of $S_A = \{Q \in SO(3) | Q\ell \in L_e\}$ are symmetric.)

To compute the index of $\ell | \mathcal{O}_{I_B}$ at $Q \in S_A$, we compute the second derivative

$$\frac{d^2}{dt^2} \ell(\exp(tW)Q)|_{t=0} = \ell(W^2 Q) .$$

Now

$$\ell(W^2 Q) = \text{trace } k(\ell, W^2 Q) = \text{trace}[k(\ell, Q) W^2]$$

$$= \text{trace } [AQ^{-1} W^2] = \text{trace } [W^2 QA]$$

because $Q^{-1} = Q$. This quadratic form on skew is represented by the element $QA - \text{trace}(QA)I$ of sym as is seen from $\hat{v}A + A\hat{v} = (Lv)^{\wedge}$ with A replaced by QA and $\text{trace}(\hat{v}\hat{w}) = 2v \cdot w$. Using the representations for $\{QA\}$ given Proposition 3.6, namely

$$\text{diag}(a,b,c),\ \text{diag}(-a,-b,c),\ \text{diag}(-a,b,-c) \text{ and } \text{diag}(a,-b,-c)$$

one checks that all four indices occur. ■

<u>Remark</u>. This lemma is a special case of the general problem to study the critical points of linear functionals on orbits of a

representation of a Lie group. This situation will arise again in our analysis of the other load types, cf. Frankel [1965] and Ramanujam [1969].

§6. Classification of Orbits in M_3

The purpose of this section is to classify orbits in M_3 under the action $(Q,A) \mapsto QA$ of $SO(3)$ on M_3 by the way the orbits meet sym. It is enough to consider orbits $\mathcal{O}_A$ of elements of sym by the polar decomposition. We begin by recalling a result already proved.

6.1. Proposition (Type 0). Suppose $A \in$ sym has no axis of equilibrium and has distinct eigenvalues. Then $\mathcal{O}_A \cap$ sym consists of four points at each of which the intersection is transversal.

This was Proposition 3.6. (Another proof of this is given below.)

We shall let the eigenvalues of $A \in$ sym be denoted a, b, c. Using the terminology from §3, we say that A has no axis of equilibrium when $(a+b)(b+c)(a+c) \neq 0$; i.e. $a + b + c \neq a$, b or c, and in this case $\mathcal{O}_A$ intersects sym transversely at A.

6.2. Definition. We shall say A is of type 1 if A has no axis of equilibrium and if exactly two of a,b,c are equal and non-zero (say $a = b \neq c$, $a \neq 0$).

6.3. Proposition. If A is type 1, then $\mathcal{O}_A \cap$ sym consists of two points (each with no axis of equilibrium) and an $\mathbb{RP}^1 \approx S^1$ (each point of which has one axis of equilibrium).

Before proving this, we give a number of lemmas of general utility. If $\ell \in \mathbb{RP}^2$ is a line through the origin in $\mathbb{R}^3$, let Q_ℓ be the rotation through angle π about ℓ.

6.4. Lemma. *$\ell \mapsto Q_\ell$ is an embedding of $\mathbb{RP}^2$ onto $SO(3) \cap \mathrm{sym} \setminus I$.*

Proof. It is clear that $\ell \mapsto Q_\ell$ is a one-to-one map of $\mathbb{RP}^2$ into $SO(3)$. Since $Q_\ell^2 = I$, $Q_\ell = Q_\ell^{-1} = Q_\ell^T$ so it maps into $SO(3) \cap \mathrm{sym}$.

Every $Q \in SO(3)\setminus I$ is a rotation through some angle θ about some axis ℓ. If such Q also is symmetric then it has three independent real eigenvectors. Hence $\theta = \pi$. ■

6.5. Corollary. *The orbit of the identity, $\mathcal{O}_I$, meets sym in one point (I) and $\mathbb{RP}^2 \cong (SO(3) \cap \mathrm{sym})\setminus I$.*

6.6. Lemma. *Let $A \in \mathrm{sym}$ with $\dim \ker A \le 1$ and suppose that $Q \in SO(3)\setminus I$ and $QA \in \mathrm{sym}$. Then $Q = Q_\ell$ for some line ℓ invariant under A, and in particular $Q \in \mathrm{sym}$.*

Proof. We can suppose $Q \neq I$. There is a unique (up to sign) unit vector $x \in \mathbb{R}^3$ such that $Qx = x$. Since $QA \in \mathrm{sym}$, we have $QA = AQ^T$, so $QAQ = A$. Thus $QAx = Ax$, so $Ax = cx$ for a constant c. Each of Q and A leaves $V = x^\perp$, the orthogonal complement of x, invariant, and A is not identically zero on V.

Let $S = QA \in \mathrm{sym}$, so $S^2 = A^2$. Since $Q|V$ is a rotation this implies $S|V = \pm A|V$ giving $Q = I$ or $Q = Q_{\ell(x)}$ where $\ell(x)$ is the line through x. Then $Q \in \mathrm{sym}$ as in Lemma 6.4. ■

It follows that if dim ker A [illegible] 1 and $QA \in$ sym, then $QA = AQ$, so as Q is both orthogonal and symmetric, A and Q can be simultaneously diagonalized.

<u>Proof of 6.1.</u> If A has distinct eigenvalues, its eigenvectors are unique, so Q is either the identity or is a rotation by π about one of the eigenvectors. ■

<u>Proof of 6.3.</u> Suppose $0 \neq a = b \neq c$, and let w be an eigenvector corresponding to the eigenvalue c. Let V be the plane orthogonal to w so V is the eigenspace with eigenvalue a. As Q, A can be simultaneously diagonalized and Q is a rotation by π (excluding $Q = I$) we have either $Q = Q_{\ell(w)}$ or $Q = Q_\ell$ for ℓ a line in V.

In the former case, $Q_{\ell(w)}A$ has eigenvalues $(-a,-a,c)$ so has no axis of equilibrium. In the latter case, $Q_\ell A$ has eigenvalues $(a,-a,-c)$, so w is an axis of equilibrium, (see 3.3). ■

6.7. <u>Corollary</u>. <u>The</u> $\mathbb{RP}^1$ <u>in Proposition 6.3 is a right coset of the subgroup</u> S^1_w <u>of all rotations about</u> w; <u>in fact</u> $\mathbb{RP}^1 = S^1_w Q_{\ell_0}$ <u>where</u> ℓ_0 <u>is any fixed line in</u> V, <u>the plane orthogonal to</u> w.

<u>Proof.</u> $\mathbb{RP}^1 = \{Q_\ell \mid \ell \in V\}$ and we have the easily verified identity

$$Q_{\ell_{\theta/2}} = \exp(\theta w)Q_{\ell_0}$$

where ℓ_θ makes an angle θ with ℓ_0 (in the positive sense) in V.

These lemmas also enable us to handle the next type.

6.8. Definition. We shall say A is of type 2 if A has no axis of equilibrium and all three of a,b,c are equal (and so $\neq 0$).

6.9. Proposition. If A is type 2, then $\mathcal{O}_A \cap$ sym consists of one point (A) and an $\mathbb{RP}^2$.

Proof. This is immediate from 6.5. ∎

Notice that each point of the $\mathbb{RP}^2$ has a whole circle of axes of equilibria; namely $Q_\ell A$ has as axes of equilibria all vectors orthogonal to ℓ. The eigenvalues of $Q_\ell A$ are $a,-a,-a$.

Types 0,1, and 2 exhaust all symmetric matrices with no axis of equilibrium, and it is easy to check from the above that any symmetric A with $\dim \ker A \leq 1$ lies on the $SO(3)$-orbit of a type 0,1, or 2 matrix. From now on we shall call these orbits, or any representatives of them, type 0,1, or 2.

Finally we turn to the remaining A's with an axis of equilibrium that is not already on an orbit of type 0,1, or 2.

6.10. Definition. We say A is type 3 if $\dim \ker A = 2$ and say A is type 4 if $A = 0$.

6.11. Proposition. If A is type 3, then $\mathcal{O}_A \cap$ sym consists of two points, A and $-A$.

Proof. $S = QA \in \text{sym}$ implies $S^2 = A^2$ and so again $S = \pm A$, as in 6.6 even though possibly $A|V = 0$. In this case Q could be any rotation about $\ell(x)$. ∎

All the foregoing information can be summarized as follows:

6.12. Theorem. The $SO(3)$ orbits in M_3 fall into five distinct types according to the way in which they meet sym (see Table 1 below). Furthermore, for $A \in \text{sym}$, $S_A = \{Q | QA \in \text{sym}\}$ consists of I $\{Q_\ell\}$ for all ℓ invariant under A (and hence $S_A \subset \text{sym}$) except in the cases (1) $\dim \ker A = 2$; then S_A also contains the rotations through any angle about the eigen-axis of A corresponding to the non-zero eigenvalue;

(2) $A = 0$; then $S_A = SO(3)$. See Table 2 below.

Remarks

1. Table 1 highlights the fact that having an axis of equilibrium or not is not an invariant of the $SO(3)$ action on L. This means there are equilibrated loads having an axis of equilibrium, but which, when rotated globally by a certain amount to another equilibrated load, no longer have one.
2. Thus, by Theorem 5.1, we get existence of solutions to the traction problem for all types of astatic loads except 3,4.*
3. The notion of type can be pulled back from M_3 to L with a little care, as we see below.

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*In particular the occurrence of 0 solutions by Stoppelli in type 1 is seen to be due to a neglect of the full rotation group (see Section 8). Our results are also consistent with those of Ball [1977].

6.13. Definition.

By analogy with our definition

$$S_A = \{Q \in SO(3) \mid QA \in \text{sym}\} \quad , \quad A \in M_3$$

which we applied when A is of type 0 , let us now write

$$S_\ell = \{Q \in SO(3) \mid Q_\ell \in L_e\} \ , \ \ell \in L \ .$$

From equivariance of k we clearly have:

6.14. Lemma. $S_\ell = S_{k(\ell)}$.

Note that the map $S_A \to \mathcal{O}_A \cap \text{sym}: Q \mapsto QA$ is an embedding for types 0,1,2 but not all types 3,4, because of the isotropy. Pulling back to L , we see that $Q \mapsto Q\ell$ is an embedding $S_\ell \to \mathcal{O}_\ell \cap L_e$ if $k(\ell)$ is of type 0,1 or 2, so we can refer to ℓ as being of type 0,1, or 2 according as $k(\ell)$ is. On the other hand, if $k(\ell)$ is of type 3 then

either (a) $\mathcal{O}_\ell \cap L_e = \{\ell, -\ell\}$

or (b) $\mathcal{O}_\ell \cap L_e =$ two disjoint circles in $\{\ell, -\ell\} + \ker k$.

Finally, if $k(\ell)$ is of type 4 then $\mathcal{O}_\ell \subset \ker k \subset L_e$ and any SO(3) orbit $\mathcal{O}_\ell$ is allowable.

TABLE 1 <u>Orbit types in</u> M_3 <u>under the</u> SO(3)-<u>action</u>.

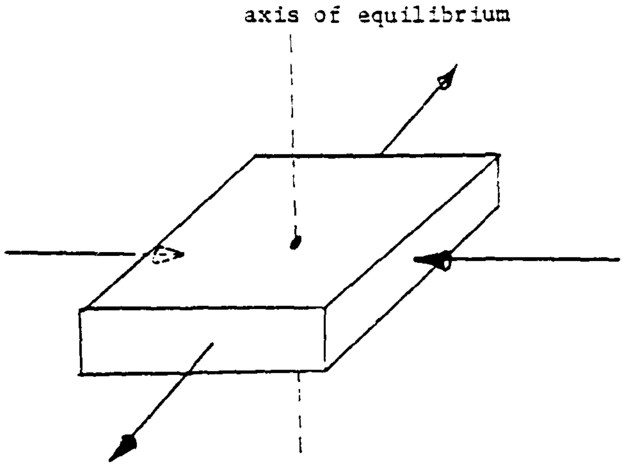
Type	Dim of Orbit	Orbit ∩ sym	Picture	Set	Eigenvalues	Isotropy	Dim orbit ∩ sym	Axes of Equilibrium
0	3	four points	x x x x	$A, \{Q_{\ell_i}\}A$ ℓ_i = eigenspaces of A	a,b,c distinct $(a+b)(a+c)(b+c) \neq 0$	none	0 $\pitchfork$	none
1	3	two points	x	A,	(a,a,c) $a \neq c$, $a \neq 0$	none	0 $\pitchfork$	none
		and	x	Q_{ℓ_1} : $\ell_1 \in 1$ dim'l eigenspace	$(-a,-a,c)$	none	0 $\pitchfork$	none
		$\mathbb{R}P^1$		$\{Q_\ell A\}$: $\ell \in 2$ dim'l eigenspace	$(a,-a,-c)$	none	1	one(ℓ_1)
2	3	one point	x	A	(a,a,a) $a \neq 0$	none	0 $\pitchfork$	none
		and $\mathbb{R}P^2$		$\{Q_\ell A\}$ arbitrary	$(a,-a,-a)$	none	2	circle (any axis orthogonal to ℓ)
3	2	two points	x	A	$(0,0,c)$ $c \neq 0$	S^1	0	circle axis orthogonal to the eigenspace of c)
			x	-A	$(0,0,-c)$	S^1	0	
4	0	one point	x	A = 0	$(0,0,0)$	SO(3)	0	$\mathbb{R}P^2$ (any axis)

$\pitchfork$ denotes transversal intersection of the orbit of A with sym.

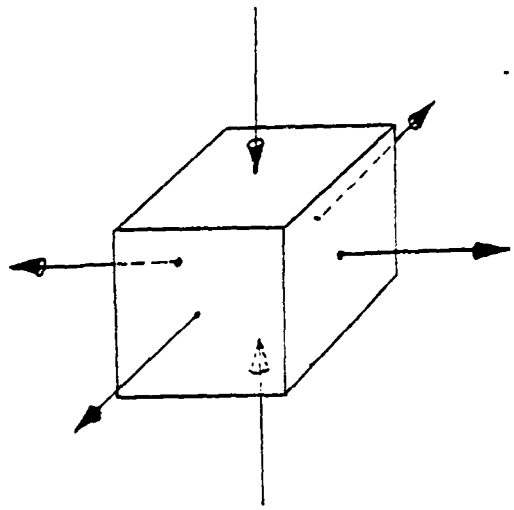
Type of Λ	Description of S_Λ
0	four points
1	two points and $\mathbb{RP}^1 \approx S^1$
2	one point and $\mathbb{RP}^2$
3	two disjoint circles
4	$SO(3)$

TABLE 2

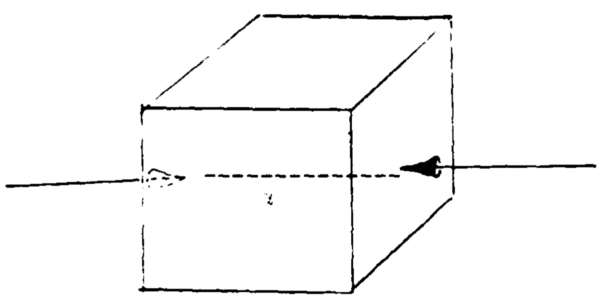
Figure 3 illustrates some simple examples of loads of different types. These loads are all pure traction, with $B = 0$.



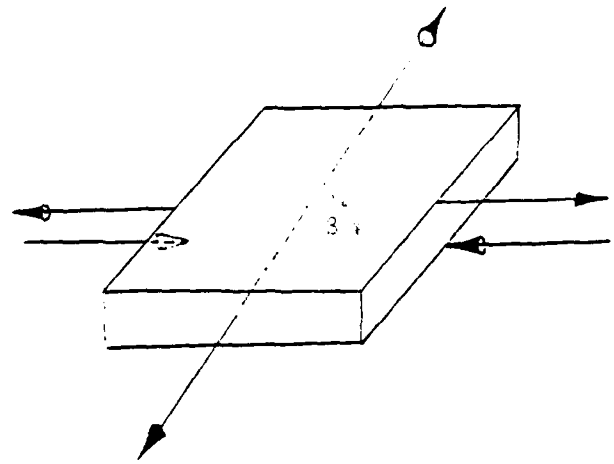
Type 1. Rotation by 180° about one of the horizontal axes produces an equilibrated load with no axis of equilibrium.



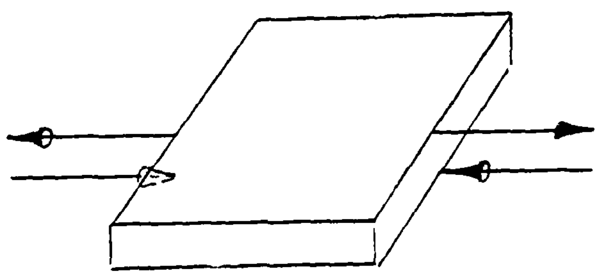
Type 2. Any horizontal axis is an axis of equilibrium; vertical axis is not an axis of equilibrium. Rotation by 180° about the vertical axis gives an equilibrated load with no axis of equilibrium.



Type 3 (a). The load itself admits a circle group of summetries about the axis ℓ-- which is thus an axis of equilibrium.



Type 3 (σ). The load is not symmetric, but the astatic load remains constant under rotation about the axis 3--which is thus an axis of equilibrium.



Type 4 . The astatic load is zero: all axes are axes of equilibrium.

Figure . Load types

§7. The Bifurcation Equation and its Gradient Character

According to the formulation (P4) of our problem, we wish to solve the equation $H(\lambda,A,n;Q) = 0$ for Q, where

$$H(\lambda,A,n;Q) \equiv \mathrm{Skew}(QA) - \lambda\bar{F}(\lambda,\mathrm{Sym}(QA),Qn),$$

(A,n) is near $(A_0,n_0) \in \mathcal{L}_e$ and λ is small. Define the right-invariant vector field X_{A_0} on $SO(3)$ by

$$X_{A_0}(Q) = \mathrm{Skew}(QA_0)\cdot Q;$$

i.e. right translation of $\mathrm{Skew}(QA_0) \in so(3) = T_I SO(3)$ to $T_Q SO(3)$. Likewise, we shall regard H as a right-invariant vector field on $SO(3)$ depending on the parameters λ,A,n by setting

$$X(\lambda,A,n;Q) = H(\lambda,A,n;Q)\cdot Q$$

Thus,

$$X(0,A_0,n_0;Q) = X_{A_0}(Q).$$

Finally, note that S_{A_0} is the zero set of X_{A_0}; i.e.,

$$S_{A_0} = \{Q \in SO(3) \mid \mathrm{Skew}(QA_0) = 0\}.$$

What S_{A_0} is for various types was given in [illegible]

7.1. <u>Lemma</u>. <u>Suppose</u> $A_0 \in \text{sym}$ <u>is of any type. Then for</u> $Q \in S_{A_0}$,

$$T_Q S_{A_0} = \{WQ \mid W \in \text{skew and } WQA_0 + QA_0W = 0\} = \ker DX_{A_0}(Q).$$

<u>Proof</u>. The second equality is clear for any A_0, because $DX_{A_0}(Q)$: $WQ \mapsto \text{skew}(WQA_0)\cdot Q$. For the first one, the inclusion $\subset$ immediately follows by differentiation of $X_{A_0}(Q) = 0$ in Q. Equality then follows by a dimension count; recall from 3.3 that $v \mapsto \hat{v}$ gives an isomorphism from the space of axes of equilibrium for A (not necessarily of unit length) to the $W \in \text{skew}$ such that $WA + AW = 0$. ■

Recall that $W \mapsto WQA_0 + (QA_0)^T W$ corresponds to the linear transformation $\text{trace}(QA_0)I - QA_0$ under the isomorphism of skew $= SO(3)$ with $\mathbb{R}^3$. When $Q \in S_{A_0}$, QA_0 is symmetric, so this transformation is symmetric relative to the Killing form on $SO(3)$. This remark and 7.1 proves the next lemma.

7.2. Lemma. Suppose λ_0 is of any type. Then at each point Q of S_{λ_0}, the range of $DX_{\lambda_0}(Q): T_Q SO(3) \to T_Q SO(3)$ is the orthogonal complement of $T_Q S_{\lambda_0}$.

Next we recall a general context for the bifurcation of vector fields that will be applied to our situation (cf. Reeken [1973]). Let M and Λ be manifolds and $X: M \times \Lambda \to TM$ a smooth vector field on M depending on the parameters $\lambda \in \Lambda$. We seek the zeros of X. For $\lambda = \lambda_0$, suppose the zero set S of X is a known smooth compact submanifold of M. Assume that M carries a Riemannian metric and that for $x \in S$, the range of $D_x X(x,\lambda_0)$ is the orthogonal complement of $T_x S$. The normal bundle E of S trivializes a neighborhood U of S. For each $x \in U$, let $P_x: T_x M \to T_x S_{\pi(x)}$ be the orthogonal projection to the fiber $S_{\pi(x)}$ over $\pi(x)$, where $\pi: E \to S$ is the projection. By the inverse function theorem, there is a unique section $\phi_\lambda: S \to E$ such that $P_x X(\phi_\lambda(x),\lambda) = 0$ for $x \in S$ and λ in a neighborhood of λ_0 (assume, for example, that S is compact). Let $\tilde{X}(x,\lambda)$ be the orthogonal projection of $X(x,\lambda)$ onto the tangent space to the graph of ϕ_λ at a point x on the graph. Thus, $\tilde{X}(x,\lambda)$ is a vector field on the graph of ϕ_λ and finding its zeros is clearly equivalent (for small λ) to finding zeros of X. We call the equation $\tilde{X}(x,\lambda) = 0$ on the graph of ϕ_λ the bifurcation equation. Since S

and the graph of ϕ_λ are diffeomorphic under π_λ , we can equally well regard $\tilde{X}$ as a vector field on S. This reduction of the problem is often known as the Liapunov-Schmidt method.

The above procedure may be applied to our vector field $X(\lambda,\Lambda,n;Q)$ with parameters (λ,Λ,n) and variable $x = Q \in SO(3) = M$. Assume λ is near zero and (Λ,n) is near a load (Λ_0,n_0) where Λ_0 is of arbitrary type. Thus, there is a unique section $\phi_{\lambda,\Lambda,n}$ of the normal bundle to S_{Λ_0} determined by the Liapunov-Schmidt procedure as described above. Let $\Gamma(\lambda,\Lambda,n)$ denote the graph of $\phi_{\lambda,\Lambda,n}$ and let $\tilde{X}(\lambda,\Lambda,n;Q)$ be the orthogonal projection of X to the tangent space of Γ at Q. Thus, $\tilde{X}$ is a vector field on Γ. As above, we may also regard $\tilde{X}$ as a vector field on S_{Λ_0}.

The rest of this section is devoted to proving that the essential part of $\tilde{X}$ is a gradient. In the general context above, if X is a gradient, then so is $\tilde{X}$ since the orthogonal projection of a gradient vector field to a submanifold is the gradient of the restriction. This simple version does not directly apply to our situation as X need not be a gradient vector field on $SO(3)$. However, the "second order" Taylor approximation $\tilde{X}_2$ of $\tilde{X}$ will be.

To state our gradient results, recall that in §4 we defined the quadratic function $G: L_e \to$ skew to be the second order term in the Taylor expansion of F about 0. Thus $\bar{F}(\lambda,\ell) = \frac{1}{2}G(\ell) + \frac{1}{6}C(\ell) + \ldots$ where G is a quadratic function of ℓ. The appropriate second order approximation to the vector field X will thus be defined by

$$X_2(\lambda,\Lambda,n;Q) = \lambda\,\mathrm{skew}(Q\Lambda) - \frac{1}{2}\mathrm{skew}\;G(Q\ell_0)\cdot n$$

Let $\tilde{X}_2$ be the second order approximation of the vector field $\tilde{X}$ on S_{A_0} obtained by the Liapunov-Schmidt procedure. Thus, $\tilde{X}_2(Q)$ is the orthogonal projection of X_2 onto the tangent space $T_Q S_{A_0}$ for $Q \in S_{A_0}$.

7.3. Theorem. *Suppose* A_0 *is of arbitrary type. Then* $\tilde{X}_2$ *is a gradient vector field on* S_{A_0}. *In fact,* $\tilde{X}_2 = -\text{grad } f$, *where*

$$f(Q) = \langle \ell_0, Q^T I_{\mathcal{B}} \rangle + \langle \ell_0, \tfrac{\lambda}{2} Q^T u_Q \rangle = \langle \ell_0, Q^T I_{\mathcal{B}} \rangle + \frac{\lambda}{2} \int_{\mathcal{B}} \langle \nabla u_Q, c(\nabla u_Q) \rangle \, dV$$

and $u_Q = D\tilde{\Phi}(I_{\mathcal{B}})^{-1}(Q\ell_0)$; *i.e.,* u_Q *is the unique solution in* $\mathcal{C}_{sym}$ *of the linearized equations with load* $Q\ell_0 \in L_e$.

Recall that the pairing between loads $\ell = (B, \tau)$ and configurations (or displacements) is given by

$$\langle \ell, \phi \rangle = \int_{\mathcal{B}} B(X) \cdot \phi(X) \, dV + \int_{\partial \mathcal{B}} \tau(X) \cdot \phi(X) \, dA = \text{trace } k(\ell, \phi)$$

and physically represents a potential for the working of the loads. Observe that if $\ell \in L_e$, then $\langle \ell, Q^T I_{\mathcal{B}} \rangle = \text{trace}(AQ) = \text{trace}(AQ^T) = \langle \ell, Q I_{\mathcal{B}} \rangle$ for all $Q \in SO(3)$.

Remark. In the second term of X_2 and f we can replace ℓ_0 by ℓ. However, the difference is higher order, so ℓ_0 is sufficient for subsequent applications.

Proof of 7.3. We shall show that X_2 is a gradient field on $SO(3)$ which, by the remarks follwoing 7.2., is sufficient.

We proceed in two parts. Let us first show that $X_A(Q)$ is the gradient of $\langle \ell, Q^T I_{\mathcal{B}} \rangle$ on all of $SO(3)$.

7.4. Lemma. *Let* $\ell \in L$ *and* $A = k(\ell)$. *Define a vector field* X_A *on* $SO(3)$ *by* $X_A(Q) = \text{Skew}(QA)\cdot Q$ *as above and the map* $\tilde{\ell}$ *of* $SO(3)$ *to* $\mathbb{R}$ *by* $\tilde{\ell}(Q) = \langle \ell, Q^T I_{\mathcal{B}} \rangle$. *Then* $X_A = -\text{grad}\, \tilde{\ell}$.

Proof. Two simple, but useful observations are:

If B, $W \in M_3$, with $W \in$ skew, then

$$\langle B, W \rangle = \langle \text{skew } B, W \rangle \tag{1}$$

and

If $B \in M_3$, $\ell \in L$ and $\phi \in C$, then

$$\langle \ell, B\phi \rangle = \langle B, k(\ell, \phi) \rangle \tag{2}$$

To prove 7.4, we compute as follows:

$$\begin{aligned}
d\tilde{\ell}(Q)\cdot(WQ) &= \langle \ell, (WQ)^T I_{\mathcal{B}} \rangle \\
&= \langle (WQ)^T, k(\ell, I_{\mathcal{B}}) \rangle \quad \text{by (2)} \\
&= \langle (WQ)^T, A \rangle \\
&= \langle W^T, QA \rangle \\
&= -\langle W, \text{skew}(QA) \rangle \quad \text{by (1)} \\
&= -\langle WQ, \text{skew}(QA)\cdot Q \rangle \\
&= -\langle WQ, X_A(Q) \rangle \quad . \quad \blacktriangledown
\end{aligned}$$

This deals with the first term of $\tilde{X}_2$. To deal with the second term, we need a special case of Betti's reciprocity theorem:

7.5. Lemma. $\langle Q\ell_0, u_{WQ}\rangle = \langle (WQ)\ell_0, u_Q\rangle$ for $Q\ell_0$ and $(WQ)\ell_0 \in \mathcal{S}ym$.

This is a trivial consequence of symmetry of $D\Phi(I_B)$ i.e. of the elasticity tensor. It is also proved in standard references; for example, see Truesdell and Noll [1965]; p. 325.

To prove 7.3., we shall also need to calculate the second derivative of the skew component of Φ; i.e. of $F(\phi) = \mathrm{Skew}[k(\Phi(\phi))]$. Surprisingly, this second derivative depends only on the classical elasticity tensor c. Recall from §2 that we regard c as a linear map of sym to itself and that we write $e = \frac{1}{2}(\nabla u + (\nabla u)^T)$.

7.6. Lemma. Let $F: \mathcal{C} \to$ skew be defined by $F(\phi) = \mathrm{Skew}[k(\Phi(\phi))]$. Then $F(I_B) = 0$, $DF(I_B) = 0$ and

$$D^2F(I_B)(u,u) = 2\ \mathrm{Skew}\left(\int_B \nabla u \cdot c(e)\,dV\right) = -2\ \mathrm{Skew}\ k(\ell_u, u)$$

where $\ell_u = (b_u, \tau_u)$, $b_u = -\mathrm{DIV}(c(e))$ and $\tau_u = c(e)\cdot N$. Identifying skew with $\mathbb{R}^3$, this becomes

$$-D^2F(I_B)(u,u) = \int_B b_u \times u\ dV + \int_{\partial B} \tau_u \times u\ dA$$

<u>Proof</u>. By Lemma 4.2, $F(\phi) = \text{Skew}[\int_{\mathcal{B}} P\,dV]$ where P is the first Piola-Kirchhoff stress tensor. We have $P(I_{\mathcal{B}}) = 0$, so $F(I_{\mathcal{B}}) = 0$. Also, $DF(I_{\mathcal{B}})\cdot u = \text{skew} \int_{\mathcal{B}} \frac{\partial P}{\partial F}\cdot \nabla u \; dV = \text{skew} \int_{\mathcal{B}} c\cdot e \; dV = 0$ as $c\cdot e$ is symmetric and since $\frac{\partial P}{\partial F}(I_{\mathcal{B}}) = c$. To compute D^2F, we shall need to use the fact that S is symmetric. Write $P = FS$ and use the product rule to obtain $D_F P(F)\cdot \nabla u = \nabla u\cdot S(F) + F D_F S(F)\cdot \nabla u$. Thus, as $S(I_{\mathcal{B}}) = 0$,

$$D^2P(I_{\mathcal{B}})\cdot(\nabla u,\nabla v) = \nabla u\cdot D_F S(I_{\mathcal{B}})\cdot \nabla v + \nabla v\cdot D_F S(F)\cdot \nabla u + D_F^2 S(I_{\mathcal{B}})\cdot(\nabla u,\nabla v).$$

Now $D_F S(I_{\mathcal{B}})\cdot \nabla u = D_C S(I_{\mathcal{B}})\cdot(\nabla u + \nabla u^T) = c\cdot e$ and $D_F^2 S(I_{\mathcal{B}})$ is symmetric, so

$$\begin{aligned} D^2F(I_{\mathcal{B}})\cdot(u,u) &= \text{skew} \int_{\mathcal{B}} D_F^2 T(I_{\mathcal{B}})(\nabla u,\nabla u)\,dV \\ &= 2\ \text{skew}\left(\int_{\mathcal{B}} \nabla u\cdot c(e)\ dV\right) \end{aligned}$$

Finally, this equals

$$-2\ \text{skew}\left\{\int_{\mathcal{B}} b_{u}\otimes u\ dV + \int_{\partial\mathcal{B}} \tau_{u}\otimes u\ dA\right\}$$

by the divergence theorem, so the last statement follows. ▼

7.7. Example. For a homogeneous isotropic material,

$$\sigma(e) = \lambda(\text{trace } e)I + 2\mu e$$

where $e = [\nabla u + (\nabla u)^T]/2$ and λ, μ are the Lamé moduli. Thus,

$$D^2F(I_{\mathcal{B}})(u,u) = 2 \text{ Skew } \left(\int_{\mathcal{B}} \{\lambda \nabla u \cdot [\text{trace } (\nabla u)]I + 2\mu \nabla u \cdot e\} dV\right.$$

$$= 2 \text{ Skew } \int_{\mathcal{B}} \{\lambda[\text{trace}(\nabla u)]\nabla u + \mu \nabla u \cdot \nabla u\} dV \quad \blacksquare$$

Let us next see what 7.6 says about the quadratic term G in the Taylor expansion of F. For $\phi \in \mathcal{C}_{sym}$ we have the identity

$$F(\phi) = F(P_e \Phi(\phi))$$

where $P_e : L \to L_e$ is the orthogonal projection. Thus, as DF and DF are zero at $I_{\mathcal{B}}$ and 0 respectively, and $P_e D\Phi(I_{\mathcal{B}}) = D\Phi(I_{\mathcal{B}})$, we get

$$D^2F(I_{\mathcal{B}})(u,v) = D^2F(0)(D\Phi(I_{\mathcal{B}})\cdot u, D\Phi(I_{\mathcal{B}})\cdot v)$$

Hence for $\ell \in L_e$ and writing $u_\ell = D\Phi(I_{\mathcal{B}})^{-1}\ell$, we have the identity

$$G(\ell) = D^2F(I_{\mathcal{B}})(u_\ell, u_\ell)$$

i.e.

$$-G(\ell) = 2 \text{ Skew } [\int_{\mathcal{B}} b \otimes u_\ell dV + \int_{\partial\mathcal{B}} \tau \otimes u_\ell dA]$$

$$= 2 \text{ Skew } k((b,\tau),u_\ell)$$

where $b = -\mathrm{DIV}(c\cdot(e)_\ell)$, and $\tau = c(e_\ell)\cdot N$ and $e_\ell = [\nabla u_\ell + (\nabla u_\ell)^T]/2$. However, these last equations say exactly that $(b,\tau) = \ell$, and so we get

$$-\frac{1}{2}G(\ell) = \text{Skew } k(\ell,u_\ell) \qquad (3)$$

Completion of the proof of 7.3. The derivative of $Q \mapsto \langle \ell_0, \frac{\lambda}{2}Q^T u_Q\rangle$ in the direction WQ is given by

$$\langle \ell_0, \frac{1}{2}(WQ)^T u_Q\rangle + \langle \ell_0, \frac{1}{2}Q^T u_{WQ}\rangle$$

$$= \langle \ell_0, (WQ)^T u_Q\rangle \qquad \text{(by Betti reciprocity, 7.5)}$$

$$= -\langle Q\ell_0, Wu_Q\rangle$$

$$= -\langle W, k(Q\ell_0, u_Q)\rangle \qquad \text{by (2)}$$

$$= -\langle W, \text{Skew } k(Q\ell_0, u_Q)\rangle \quad \text{by (1)}$$

$$= -\langle WQ, \text{Skew } k(Q\ell_0, u_Q)Q\rangle$$

$$= \langle WQ, \frac{1}{2}G(Q\ell_0)\cdot Q\rangle \qquad \text{by (3)} . \blacksquare$$

§8. Bifurcation Analysis for Type I

We now discuss the solutions of the basic equation

$$H(\lambda,A,N;Q) = \text{Skew}(QA) - \lambda\bar{F}(\lambda,\text{Sym}(QA),Qn) = 0 \qquad (1)$$

for the load $\ell = (A,n)$ near a load $\ell_0 = (A_0,n_0)$ having an axis of equilibrium and of type 1, and for λ near 0. We shall also obtain the stability of the solutions and finally we shall compare our results with those of Stoppelli [1958]. For type 1 we need to do a bifurcation analysis on the circle S_{A_0} corresponding to the degenerate zero set of H when $\lambda = 0$ and $\ell = \ell_0$. The analysis has some features in common with the papers of Hale [1977] and Hale and Taboas [1981].

Without loss of generality we can assume that $A_0 = \text{diag}(a,-a,-c)$ where $0 \neq a^2 \neq c^2$. Thus, from §6 the set S_{A_0} of zeros of Skew (QA) for $Q \in SO(3)$ is given explicitly by the following two points and circle:

$$S_{A_0} = \{\text{diag}(1,-1,-1)\ \text{diag}(-1,1,-1)\} \cup C_{A_0} \qquad (2)$$

where

$$C_{A_0} = \left\{ Q = \begin{pmatrix} x & -y & 0 \\ y & x & 0 \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, x = \cos\theta,\ y = \sin\theta \right\}$$

The loads corresponding to the two points are $A_0^* = \text{diag}\,(a,a,c)$ and $A_0^{**} = \text{diag}(-a,-a,c)$.

From 7.3, we are led to study the critical points of $f(Q) = \langle \ell, Q^T l_5\rangle + \frac{1}{2}\langle \ell_0, Q^T u_0\rangle$ on C_{A_0}. Note from the divergence theorem that

$$\langle \ell_Q, Q^t u_Q \rangle = \int_B \langle \nabla u_Q, c(e_Q) \rangle \, dV \tag{3}$$

where $u_Q = D\tilde{\Phi}(I)^{-1}(Q\ell_Q)$ and $e_Q = [\nabla u_Q + (\nabla u_Q)^T]/2$. Thus the function f is computable from linearized elasticity alone, which leads to the curious observation that our "second order" nonlinear elasticity here involves no more data than linear elasticity, but merely processes the information in a different way. Writing $Q = \begin{pmatrix} x & -y & 0 \\ y & x & 0 \\ 0 & 0 & 1 \end{pmatrix}$ as in (2), f becomes a polynomial of degree 2 in (x,y). Write the two terms of f as

$$f(Q) = f(x,y) = (b_0 + b_1 x + b_2 y) + \frac{\lambda}{2}(a_1 x^2 + a_2 xy + a_3 y^2 + a_4 x + a_5 y + a_6) \tag{4}$$

which defines the numerical constants b_0, b_1, b_2 and $a_1, \ldots, a_6$. Next, define new parameters $\alpha_1, \ldots, \alpha_6$ by writing

$$f^*(x,y) = \frac{2}{\lambda} f(x,y)$$

and letting

$$f^*(x,y) = \alpha_1 x^2 + \alpha_2 xy + \alpha_3 y^2 + \alpha_4 x + \alpha_5 y + \alpha_6 \tag{5}$$

Note that $\alpha_1, \ldots, \alpha_6$ depend on our parameters λ, ℓ as well as on the elastic moduli of the material. Thus,

$$\left.\begin{aligned} &\alpha_1 = a_1, \quad \alpha_2 = a_2, \quad \alpha_3 = a_3 \\ &\alpha_4 = \frac{2}{\lambda} b_1 + a_4, \quad \alpha_5 = \frac{2}{\lambda} b_2 + a_5, \quad \alpha_6 = \frac{2}{\lambda} b_0 + a_6 \end{aligned}\right\} \tag{6}$$

Replacing ℓ_0 by $Q\ell_0$, where Q is as in (2), effects a rotation of the x-y plane. Thus, by rotation of ℓ_0 if necessary, we can assume $\alpha_2 = 0$.

Let us fix α_1, α_3 and consider the bifurcations of zeros of $\frac{df^*}{d\theta} = 2(\alpha_3-\alpha_1)xy + \alpha_5 x - \alpha_4 y$ on S^1 (i.e., critical points of f^* on S^1) with α_4 and α_5 as parameters.

Set $M = \{(\alpha_4,\alpha_5,\theta) \in \mathbb{R}^2 \times S^1 : \frac{df^*}{d\theta}(\alpha_4,\alpha_5,\theta) = 0\}$, the manifold of critical points of f^*. Indeed, M is a manifold and can be parametrized by φ: $\mathbb{R} \times S^1 \to M$, $\varphi(u,\theta) = (-2(\alpha_1+u)\cos\theta, -2(\alpha_3+u)\sin\theta, \theta)$. Denote by π: $\mathbb{R}^2 \times S^1 \to \mathbb{R}^2$ the projection onto the first factor.

8.1. <u>Lemma</u>. <u>Set</u> $\Delta = [2(\alpha_1-\alpha_3)^2 - \alpha_a{}^2 - \alpha_5{}^2]^3 - 108\alpha_4{}^2\alpha_5{}^2(\alpha_1-\alpha_3)^2$. (7)

<u>If</u> $\alpha_1 - \alpha_3 \neq 0$, <u>then</u> $\pi: M \to \mathbb{R}^2$ <u>is a proper stable map in</u> (α_4,α_5)<u>-space, and its set of critical values is the astroid defined by</u> $\Delta = 0$ <u>(see Figure [illegible] below)</u>.

Since the number of points in $\pi^{-1}(\alpha)$ (i.e., the zeros of $\frac{df^*}{d\theta}$ at $\alpha = (\alpha_4,\alpha_5)$) is a constant over $\Delta < 0$ or $\Delta > 0$, we obtain

8.2. <u>Corollary</u>. $\frac{df^*}{d\theta}$ <u>has 4 zeros if</u> $\Delta < 0$, <u>and has 2 zeros if</u> $\Delta > 0$.

Proof of Lemma 3.1.

The critical set Σ of $\pi\circ\rho\colon \mathbb{R} \times S^1 \to \mathbb{R}^2$ is $\{(\mu,\theta) \in \mathbb{R} \times S^1 \mid \alpha_1 \sin^2\theta + \alpha_3 \cos^2\theta + \mu = 0\}$. Thus, the set of critical values of π can be parametrized by

$$\begin{cases} \alpha_4 = -2(\alpha_1-\alpha_3)\cos^3\theta \\ \alpha_5 = 2(\alpha_1-\alpha_3)\sin^3\theta \end{cases} \tag{8}$$

Since Σ consists of 4 cusp points, 4 fold lines and $\pi\circ\rho|\Sigma$ is 1-1 , by a result of Whitney (see Mather [1969] or Golubitsky and Guillemin [1973]), one knows that $\pi\circ\rho$ is a stable map.

Eliminating θ produces the bifurcation set

$$[2(\alpha_1-\alpha_3)]^{2/3} = \alpha_4^{\,2/3} + \alpha_5^{\,2/3} \tag{9}$$

For $\alpha_1 - \alpha_3 \neq 0$, (8) describes the astroid shown in Figure 4.

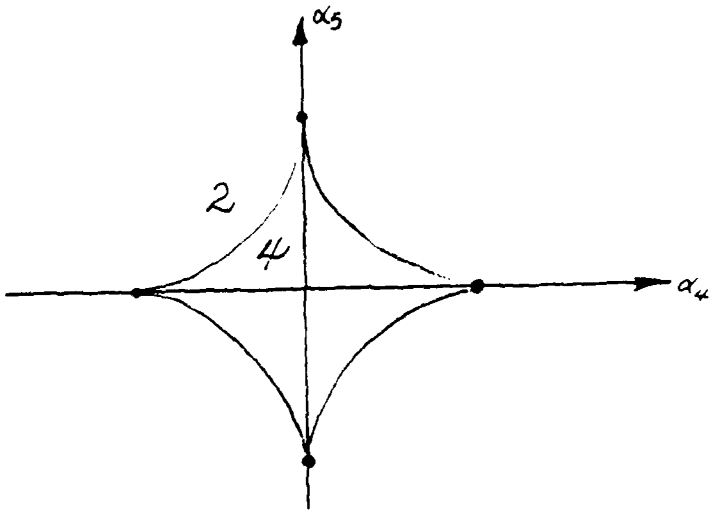


Figure 4

Next, observe that for real numbers A, B and C,

$$A + B + C = 0 \text{ if and only if } A^3 + B^3 + C^3 = 3ABC \tag{10}$$

by virtue of the identity $A^3 + B^3 + C^3 - 3ABC = (A + B + C)(A^2 + B^2 + C^2 - AB - BC - CA)$. Applying (10) to (9) shows that (9) is equivalent to $\alpha_4^2 + \alpha_5^2 - 2(\alpha_1 - \alpha_3)^2 = -3\alpha_4^{2/3}\alpha_5^{2/3}(2(\alpha_1 - \alpha_3))^{2/3}$. Cubing both sides gives the stated conclusion. ■

The family $\frac{df^*}{d\theta}$ of functions on S^1 with parameters α_4, α_5 enjoys a universal property. Consider a perturbed family $\frac{df^*}{d\theta} + g(\lambda,c,\theta)$, with $g(0,0,\theta) = 0$ for $(\lambda,c) \in \mathbb{R} \times \mathbb{R}^m$. To each (λ,c), denote by $M_{\lambda,c} = \{(\alpha_4,\alpha_5,\theta) \mid (\frac{df^*}{d\theta} + g)(\lambda,c,\alpha_4,\alpha_5,\theta) = 0\}$ the "manifold" of zeros.

8.3. <u>Lemma</u>. <u>For</u> (λ,c) <u>sufficiently small, the sets</u> $M_{\lambda,c}$ <u>are manifolds and there exist two smooth families of diffeomorphisms</u> $\varphi_{\lambda,c}: \mathbb{R}^2 \to \mathbb{R}^2$ <u>and</u> $\Psi_{\lambda,c}: M_{\lambda,c} \to M$ <u>defined for</u> λ,c <u>sufficiently small, such that</u> $\pi\circ\Psi_{\lambda,c} = \varphi_{\lambda,c}\circ\pi$, <u>and</u> $\Psi_{0,0}$ = identity, $\varphi_{0,0}$ = identity.

<u>Proof</u>: For λ,c sufficiently small, the map $\rho_{\lambda,c}: \mathbb{R}^1 \times S^1 \to M_{\lambda,c}$, $\rho_{\lambda,c}(\mu,\theta) = (-2(\alpha_1+\mu)\cos\theta + \sin\theta\,\frac{\partial g}{\partial\theta}, -2(\alpha_3+\mu)\sin\theta - \cos\theta\,\frac{\partial g}{\partial\theta}, \theta)$ defines a parametrization of $M_{\lambda,c}$. By Lemma 8.1, $\pi\circ\rho_{\lambda,c}: \mathbb{R}\times S^1 \to \mathbb{R}^2$ is an unfolding of the proper stable map $\pi\circ\rho$. Thus, one can find diffeomorphisms $\psi^*_{\lambda,c}$, $\varphi_{\lambda,c}$ on $\mathbb{R}\times S^1$ and $\mathbb{R}^2$ respectively such that $\varphi_{\lambda,c}(\pi\circ\rho_{\lambda,c}) = (\pi\circ\rho)\circ\psi^*_{\lambda,c}$. This lemma follows by letting $\Psi_{\lambda,c} = \rho\circ\psi^*_{\lambda,c}\circ\rho^{-1}_{\lambda,c}$. ■

Now, we are ready to state our main result on the number of solutions near type I loads. Let $\ell = \ell(c)$ depend smoothly on a parameter c in $\mathbb{R}^m$, with $\ell(0) = \ell_0$. Recall that Δ is defined by (7), a_1, a_2 and a_3 by (1).

8.4. <u>Theorem. Let</u> ℓ_0 <u>be a type 1 load with</u> $k(\ell_0) = (a,-a,-c)$, $0 \neq a^2 \neq c^2$, $a_2 = 0$, <u>and</u> $a_1 \neq a_3$. <u>Then, there exists a (smooth) function</u> $\tilde{\Sigma}(\lambda,c)$, $\tilde{\Sigma}(\lambda,0) = \Delta(a_4,a_5) + O(\lambda)$ <u>defined for</u> (λ,c) <u>sufficiently small and</u> $\lambda > 0$ <u>such that, the traction problem has:</u>

(i) <u>four solutions for the load</u> $\lambda\ell(c)$ <u>if</u> $\tilde{\Sigma}(\lambda,c) < 0$ <u>(two of them near</u> C_{Λ_0}) .

(ii) <u>six solutions for the load</u> $\lambda\ell(c)$ <u>if</u> $\tilde{\Sigma}(\lambda,c) > 0$ <u>(two of them near</u> C_{Λ_0}) .

<u>Proof</u>: The bifurcation of zeros of $\tilde{\lambda}$ (cf. §7) on C_{Λ_0} $(=S^1)$ is the same as finding zeros of $\langle -\tilde{\lambda}, \frac{\partial}{\partial\theta} \rangle = \langle -\tilde{\lambda}_2, \frac{\partial}{\partial\theta} \rangle + \frac{\lambda}{2} g(\lambda,c,\theta)$ $= \frac{df}{d\theta} + \frac{\lambda}{2} g$ or $\frac{df^*}{d\theta} + g$, where $g(0,0,\theta) = 0$. Let $\psi_{\lambda,c}$ be the family of diffeomorphisms found in Lemma 8.3. Take $\tilde{\Sigma}(\lambda,c) = \Delta\circ\psi_{\lambda,c}\left(\frac{b_1(\ell(c))}{2\lambda} + a_4 , \frac{b_2(\ell(c))}{2\lambda} + a_5\right)$ which has the desired property. ∎

Next, we want to determine the "generic" structure of the bifurcation set $K = \{\tilde{\lambda} = 0\}$ in (λ,c) space, $\lambda > 0$.

For $m = 0$, $\Delta(\alpha_4,\alpha_5) = \Delta(a_4,a_5) \neq 0$, it is clear that $K = \emptyset$. Indeed, our traction problem has two solutions near C_{Λ_0} if $\Delta(a_4,a_5) < 0$, and four solutions near C_{Λ_0} if $\Delta(a_4,a_5) > 0$.

For $m = 1$, consider $\tilde{k}$: $\tilde{c} \mapsto \left(\frac{b_1\Lambda(\tilde{c})}{2} + a_4, \frac{b_2\Lambda(\tilde{c})}{2} + a_5\right)$ where $\Lambda(\tilde{c})$ is the linear part of $\ell(\tilde{c})$. This represents a straight line which we assume to intersect the astroid transversely if they meet. Notice that $K_\lambda = \{c \mid (\lambda, c) \in K\}$ is the inverse image of the astroid (defined by equation (9)), under the map $h_\lambda :\ c \mapsto \varphi_{\lambda,c}\left(\frac{b_1 \ell(c)}{2\lambda} + a_4,\ \frac{b_2 \ell(c)}{2\lambda} + a_5\right)$. ($\varphi_{\lambda,c}$ as in the proof of Theorem 8.4). Write $\lambda\tilde{c} = c$. Recall that $\ell(c) = A(c) + O(|c|^2)$, and consider the map

$$\tilde{h}_\lambda \ :\ \tilde{c} \mapsto h_\lambda(\lambda, \tilde{c}) = \varphi_{\lambda,\lambda\tilde{c}}\left(\frac{b_1 A(\tilde{c})}{2} + a_4 + O(\lambda),\ \frac{b_2 A(\tilde{c})}{2} + a_5 + O(\lambda)\right) .$$

Since the astroid is bounded and $\varphi_{\lambda,c}$ is close to the identity, there exists an interval $(-M,M)$ such that $\tilde{K}_\lambda = \{\tilde{c} \mid \lambda c \in K_\lambda\} \subset (-M,M)$ for $\lambda > 0$, c sufficiently small. Applying the isotopy theorem for transversal maps (see e.g., Hirsch [1976]) to the family $\tilde{h}_\lambda$ through $\tilde{h}_0 = \tilde{k}$, we conclude that the bifurcation set $\mathcal{K}$ consists of 0, 2, or 4 curves with slopes $\tilde{k}^{-1}$(astroid) (see Figures 5,6). Thus, for example, by choosing $c \neq 0$ sufficiently small, and letting $\lambda \to 0$ (i.e., consider the load $\lambda\ell(c)$), one can pass from a parameter region where there are two solutions near the circle (four in all) to one where there are four near the circle (six in all) and back again to the two-solution region (see Figure 5).
Such a situation is not dealt with in the analysis of Stoppelli [1958].

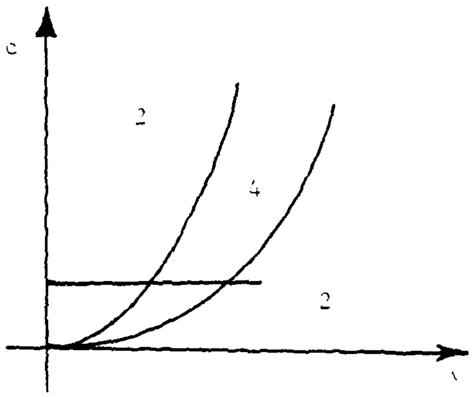


Figure 2

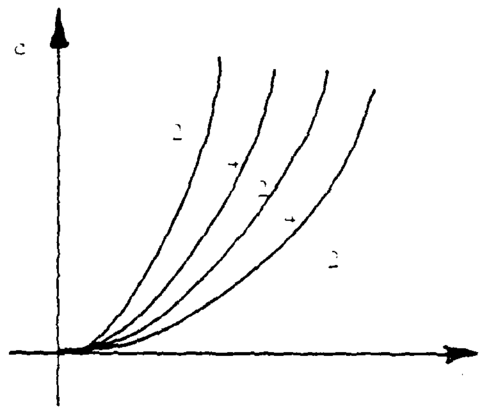


Figure 3

For $m \geq 2$, let us suppose that the affine map:
$\tilde{c} \mapsto \left(\frac{b_1A(\tilde{c})}{2} + a_4 , \frac{b_2A(\tilde{c})}{2} + a_5\right)$ is onto, where again $A(\tilde{c})$ is linear part of $\alpha(\tilde{c})$. Without loss of generality, we may also assume that $b_1A(c) = c_1$ and $b_2A(c) = c_2$, where $c = (c_1, c_2, z)$.

Notice that $K_{\lambda,z} = \{(c_1,c_2) \mid (\lambda,c_1,c_2,z) \in \mathcal{K}\}$ is the inverse image of the astroid under the map $h_{\lambda,z}\colon (c_1,c_2) \mapsto \Phi_{\lambda,c_1,c_2,z}\left(\frac{b_1A(\tilde{c})}{2\lambda} + a_4 , \frac{b_2A(\tilde{c})}{2\lambda} + a_5\right) = \Phi_{\lambda,c_1,c_2,z}\left(\frac{c_1}{2\lambda} + a_4 , \frac{c_2}{2\lambda} + a_5\right)$. Set $\lambda\tilde{c}_1 = c_1$, $\lambda\tilde{c}_2 = c_2$, and consider the map

$$\tilde{h}_{\lambda,z}: \quad (\tilde{c}_1,\tilde{c}_2) \mapsto h_{\lambda,z}(\lambda\tilde{c}_1,\lambda\tilde{c}_2) = \lambda_{\lambda,z}(\lambda\tilde{c}_1,\lambda\tilde{c}_2,z)\left(\frac{\tilde{c}_1}{2} - a_4, \frac{\tilde{c}_2}{2} + a_5\right) .$$

As before, $\tilde{K}_{\lambda,z} = \{(\tilde{c}_1,\tilde{c}_2) \mid (\lambda\tilde{c}_1,\lambda\tilde{c}_2) \in K_{\lambda,z}\}$ is bounded uniformly for $\lambda > 0$, z sufficiently small. Applying the isotopy theorem for transversal maps to the family $\tilde{h}_{\lambda,z}$ through the affine isomorphism $\tilde{h}_{0,0}$ we conclude that the bifurcation set is a cylinder-like set along the z-axis with base a cone over the astroid in c_1,c_2 space. The first order approximation of this cone is given by the cone over the astroid in the plane $\lambda = 1$, centered at $(-2a_4, -2a_5)$ with "size" $4|a_1-a_3|$ (see Figure 7).

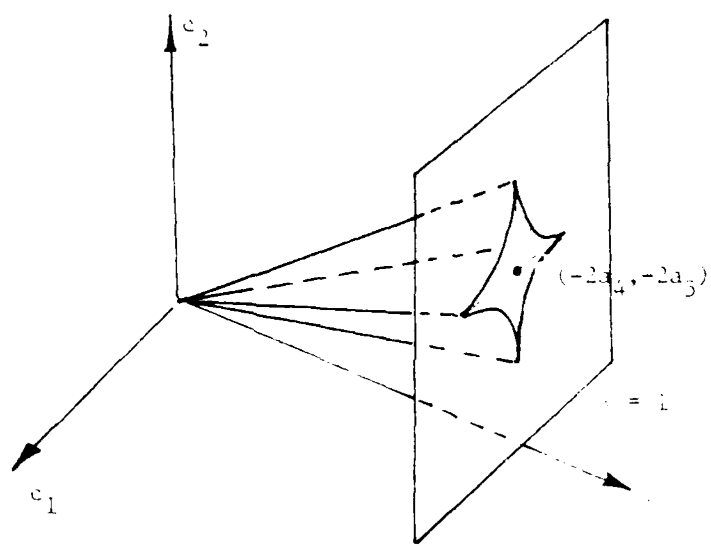


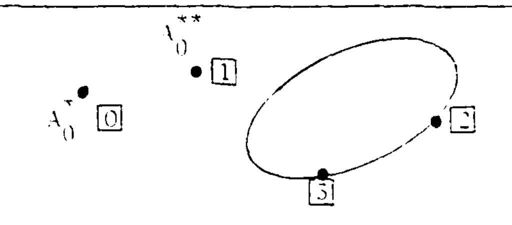
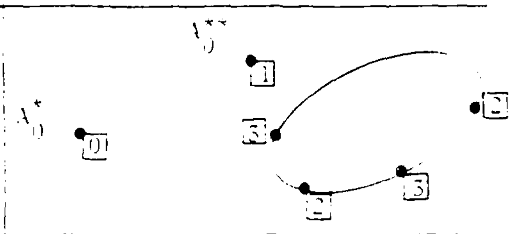
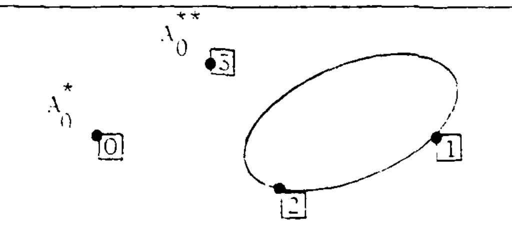
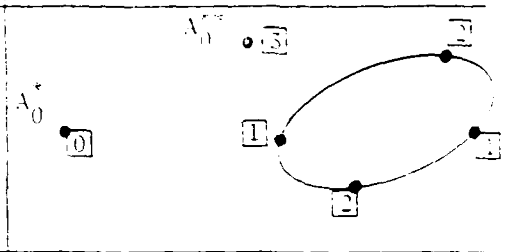
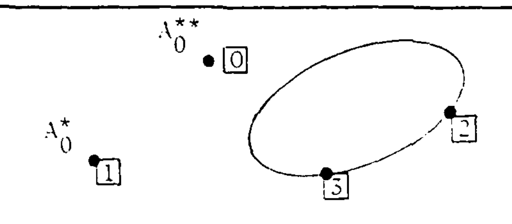
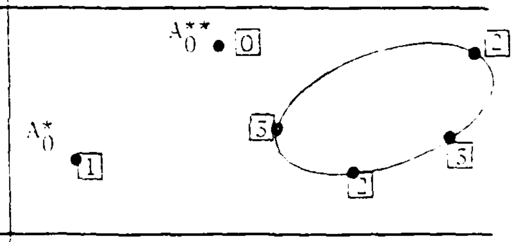
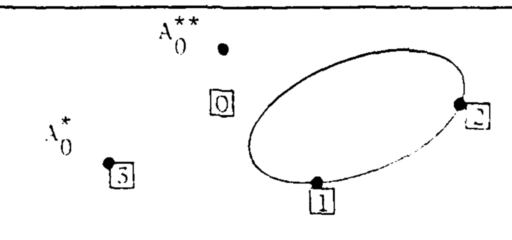
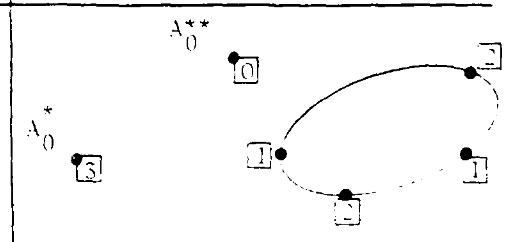
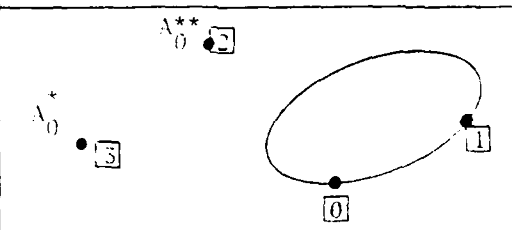
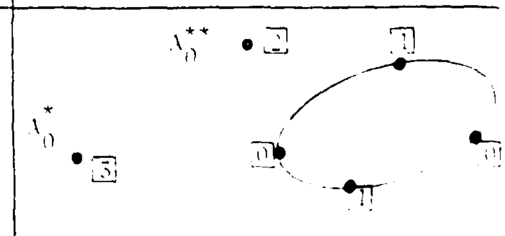
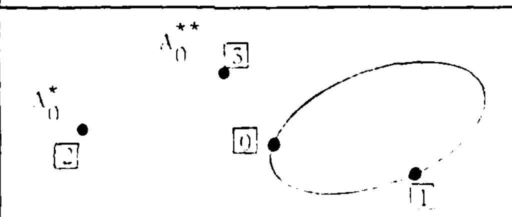
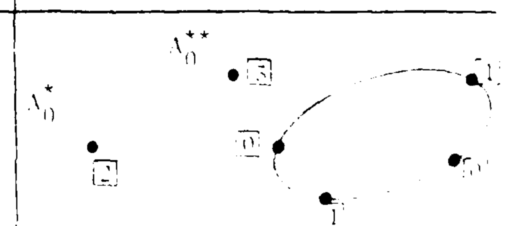
Figure 7

Next we discuss the stability of the solutions corresponding to loads near a load of type 1. This can be determined by combining our stability results for type 0 (Theorem 5.3) together with well-known stability results for the cusp. We make the same assumptions as those in Theorem 8.4.

8.5. Theorem. Let $A_0 = \mathrm{diag}(a,-a,-c)$, $A_0^* = \mathrm{diag}(a,a,c)$ and $A_0^{**} = \mathrm{diag}(-a,-a,c)$ as above. The indices of the bifurcating solutions are boxed in Table 3. (Recall that stable solutions have index = 0.) In each case the circle represents C_{A_0} defined by equation (2).

Note that two stable solutions bifurcate off the circle when $c > |a|$. In all other cases the solutions near the circle are unstable.

TABLE 3

Values of c, a	Values of $\tilde{\Sigma}$ [see Theorem 8.4]	
	$\tilde{\Sigma} < 0$	$\tilde{\Sigma} > 0$
$c < \|a\|, a < 0$ $c < a$		
$c < \|a\|$, $a < 0$ $c > a$		
$c < \|a\|$ $a > 0$ $c < -a$		
$c < \|a\|$ $a > 0$ $c > -a$		
$c > \|a\|$ $a > 0$		
$c > \|a\|$ $a < 0$		

8.4. <u>Example</u>. Let $\mathcal{B} \subset \mathbb{R}^3$ be a region with unit volume and let the load be given by $\ell_0 = (0, \tau_0)$ where

$$\tau_0 = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -c \end{pmatrix} N \quad , \qquad 0 \neq a^2 \neq c^2,$$

where N is the unit outward normal on $\partial\mathcal{B}$. Consider a homogeneous isotropic hyperelastic material whose linearized elasticity tensor c has Lamé moduli λ, μ, (see the remarks following 7.5) and is stable and strongly elliptic; i.e., $\mu > 0$, $2\mu + 3\lambda > 0$.

Thus, $k(\ell_0) = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -c \end{pmatrix}$ by the divergence theorem, and so ℓ_0 is a type 1 load. It is easy to check that

$$u_Q(X) = c^{-1}\left[Q\begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -c \end{pmatrix}\right]X \quad = c^{-1}\begin{bmatrix} ax & ay & 0 \\ ay & -ax & 0 \\ 0 & 0 & -c \end{bmatrix} X, \quad \text{for } Q = \begin{pmatrix} x & -y & 0 \\ y & x & 0 \\ 0 & 0 & 1 \end{pmatrix} \; .$$

$x^2 + y^2 = 1$, where $c^{-1}(F) = \frac{F}{2\mu} - (\text{trace } F)\frac{\lambda I}{2\mu(2\mu+3\lambda)}$. Hence,

$$\langle \ell_0, Q^T u_Q \rangle = \int_{\mathcal{B}} \langle \nabla u_Q, c(\nabla u_Q) \rangle \, dV \qquad \text{(by 7.5)}$$

$$= \left\langle c^{-1}Q\begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -c \end{pmatrix}, \; Q\begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -c \end{pmatrix} \right\rangle$$

$$= \left\langle \frac{Q\begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -c \end{pmatrix}}{2\mu} + \frac{c\lambda I}{2\mu(2\mu+3\lambda)}, \; Q\begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -c \end{pmatrix} \right\rangle$$

$$= \frac{2a^2+c^2}{2\mu} - \frac{c^2\lambda}{2\mu(2\mu+3\lambda)} \; .$$

which is a constant (independent of x, y). In this situation, $\alpha_1 = \alpha_2 = 0$ and so our theorems do not apply.

8.5. <u>Example</u>. Consider the same traction problem as above, but with a homogeneous nonisotropic hyperelastic material whose linearized elasticity tensor is given by $c(\epsilon) = \epsilon - \frac{1}{2}\operatorname{diag}\epsilon$.

In this case, $u_Q(X) = c^{-1}\left[Q\begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -c \end{pmatrix}\right] X$, $Q = \begin{pmatrix} x & -y & 0 \\ y & x & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

where $c^{-1}(F) = F + \operatorname{diag} F$. Then

$$\langle \ell_0, Q^T u_Q \rangle = \int_B \langle \nabla u_Q, c(\nabla u_Q) \rangle \, dV$$

$$= \left\langle \begin{pmatrix} 2ax & ay & 0 \\ ay & -2ax & 0 \\ 0 & 0 & -2c \end{pmatrix}, \begin{pmatrix} ax & ay & 0 \\ ay & -ax & 0 \\ 0 & 0 & -c \end{pmatrix} \right\rangle$$

$$= 4a^2x^2 + 2a^2y^2 + 2c^2 .$$

Hence, $\Delta = 8a^{12} > 0$, and our traction problem $\text{fn}(\ell_0)$ has six solutions (four near C_{A_0}), with stability determined by table 3.

Next we shall discuss how to obtain the results of Stoppelli [1958] as a special case of our analysis. We refer the reader to the statements of Stoppelli's results in Grioli [1962, p. 58]. In this approach one focuses attention on bifurcations that occur on the circle by examining what happens at a particular location on the circle and $\lambda = \lambda_0$. We can assume this point is $(1,0)$ i.e. $\theta = 0$ with no loss of generality.

First of all, if $\alpha_2 + \alpha_5 \neq 0$ then $(1,0)$ is not a critical point of f^*, so there are no solutions near $(1,0)$. We may assume then that $\alpha_2 + \alpha_5 = 0$ and then the Taylor expansion of f^* about $\theta = 0$ becomes

$$f^*(\theta) = (\alpha_1 + \alpha_6) + \left(-\alpha_1 + \alpha_3 - \frac{\alpha_4}{2}\right)\theta^2 - \frac{\alpha_2}{2}\theta^3 + \frac{1}{3}\left(\alpha_1 - \alpha_3 + \frac{\alpha_4}{8}\right)\theta^4 + (\text{higher order terms})$$

For critical points, we are seeking zeros of

$$\frac{df^*}{d\theta} = 2\left(-\alpha_1 + \alpha_3 - \frac{\alpha_4}{2}\right)\theta - \frac{3}{2}\alpha_2\theta^2 + \frac{4}{3}\left(\alpha_1 - \alpha_3 + \frac{\alpha_4}{8}\right)\theta^3 + O(\theta^4)$$

<u>Case 1</u>. If $-\alpha_1 + \alpha_3 - \frac{\alpha_4}{2} \neq 0$, then $\frac{df^*}{d\theta} = 2\left(-\alpha_1 + \alpha_3 - \frac{\alpha_4}{2}\right)\theta + O(\theta^2)$ and so there is *just one* solution. This is Theorem F on p. 58 of Grioli [1962].

<u>Case 2</u>. If $-\alpha_1 + \alpha_3 - \frac{\alpha_4}{2} = 0$ and $\alpha_2 \neq 0$, then

$\frac{df^*}{d\theta} = -\frac{3}{2}\alpha_2\theta^2 + O(\theta^3)$ and so there are 0, 1 or 2 solutions (fold point). This is Theorem G on p. 58 of Grioli [1962].

<u>Case 3</u>. If $-\alpha_1 + \alpha_3 - \frac{\alpha_4}{2} = 0$, $\alpha_2 = 0$ but $\alpha_1 - \alpha_3 + \frac{\alpha_4}{8} \neq 0$

then $\frac{df^*}{d\theta} = \frac{4}{3}(\alpha_1 - \alpha_3 + \frac{\alpha_4}{8})\theta^3 + O(\theta^4)$, so there are 1, 2 or 3 solutions (cusp point). This is Theorem H on p. 58 of Grioli [1962].

Furthermore, if we express our constants α_i $(=a_i)$ in terms of the elasticity tensor c and solutions of the linearized problem using (3) above, we find the same conditions for these three cases as is given on p. 57 of Grioli [1962].

Thus we recover the results of Stoppelli on type 1 loads. As was explained in the introduction, however, this analysis is only local on the circle and does not give the full story of the bifurcation picture, even in this case. The complete bifurcation analysis, including stability, is summarized by our Figure 7 and Table 3.

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